Real intersection theory II

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Abstract

Continuing from Part I, we explore the properties of current's intersection $[\cdot,\cdot]$ and show it is the extension of the geometric intersections in topology, differential geometry and algebraic geometry.

Contents

Before we get into the main development, we introduce the organization of part II as follows. In section 1, we prove properties of the intersection of currents. In section 2, we establish the connection between our current's intersection and geometric intersections in classical theories. In section 3, we use the current's intersection to develop further operators on currents. It leads a categorical environment where the application will arise.

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1 Property

1.1 Basic properties

Lemma 1.1. *Let $\mathcal{X}$ be a $\mathbb{C}^\infty$ manifold, and $\mathcal{Z} \subset \mathcal{X}$ a submanifold. Let*

$$\mathcal{Z} \overset{i}{\hookrightarrow} \mathcal{X}$$

be the inclusion map. Let

$$\mathscr{D}(\mathcal{X}, \mathcal{Z}) = \{\phi \in \mathscr{D}(\mathcal{X}) : \phi|_{\mathcal{Z}} = 0\}, \tag{1.1}$$

where $\phi|_{\mathcal{Z}}$ is the pullback of the C^∞-differential form by the inclusion map. So

$$\mathscr{D}(\mathcal{X}, \mathcal{Z}) \subset \mathscr{D}(\mathcal{X}). \tag{1.2}$$

Then the sequence

$$0 \quad \rightarrow \quad \mathscr{D}'(\mathcal{Z}) \quad \overset{i_*}{\rightarrow} \quad \mathscr{D}'(\mathcal{X}) \quad \overset{R}{\rightarrow} \quad \mathscr{D}'(\mathcal{X}, \mathcal{Z}) \tag{1.3}$$

is exact, where $'$ stands for the topological dual and R is the restriction map through (1.2).

Proof. Let $T \in \mathscr{D}'(\mathcal{Z})$ such that $i_*(T) = 0$. Let $q \in supp(T) \subset \mathcal{Z}$. Let $\phi \in \mathscr{D}(\mathcal{Z})$ supported in an Euclidean neighborhood of q inside of $\mathcal{Z}$. Then since $\mathcal{Z}$ is a manifold, ϕ can be extended to a C^∞ form ϕ' in a neighborhood of q inside of X. Then $\int_{i_*(T)} \phi' = \int_T \phi$ is equal to 0. Thus $T = 0$, and further i_* is injective. Next we focus on R. We assume $\mathcal{Z}$ is compact. It is trivial that $R \circ i_* = 0$. Let's show

$$ker(R) \subset Im(i_*).$$

Let U be a tubular neighborhood of $\mathcal{Z}$ and $j : U \to \mathcal{Z}$ be a projection induced from the normal bundle structure of U. Let h be a C^∞ function on $\mathcal{X}$ such that it has a compact support in U and it is 1 on $\mathcal{Z}$. For any $T \in \mathscr{D}'(\mathcal{X})$, we define a current T' on $\mathcal{Z}$

$$\int_{T'} (\cdot) := \int_T hj^*(\cdot). \tag{1.4}$$

Let $T \in ker(R)$. We would like to show

$$i_*(T') = T.$$

It suffices to show that for any testing form of ϕ on $\mathcal{X}$

$$\int_T hj^*(\phi|_{\mathcal{Z}}) = \int_T \phi,$$

or

$$\int_T \left(hj^*(\phi|_{\mathcal{Z}}) - \phi \right) = 0. \tag{1.5}$$

Since $hj^*(\phi|_{\mathcal{Z}}) - \phi$ vanishes on $\mathcal{Z}$,

$$ker(R) \subset Im(i_*),$$

so (1.3) is exact. If $\mathcal{Z}$ is non-compact, we can use a partition of unity to have the same proof. We complete the proof.

□

Proposition 1.2. *Let $\mathcal{X}$ be a manifold endowed with a de Rham data. Let $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ be a submanifold. Let $T \in C(\mathcal{X})$. Then there is a unique current denoted by $[\mathcal{Z} \wedge T]_{\mathcal{Z}}$ in $\mathcal{Z}$ such that*

$$i_*([\mathcal{Z} \wedge T]_{\mathcal{Z}}) = [\mathcal{Z} \wedge T], \tag{1.6}$$

where the intersection current $[\mathcal{Z} \wedge T]$ is defined in [6].

Proof. For any $\phi \in \mathscr{D}(\mathcal{X}, \mathcal{Z})$,

$$\int_{[\mathcal{Z} \wedge T]} \phi = \lim_{\epsilon \to 0} \int_{\mathcal{Z}} R_\epsilon^{\mathcal{X}}(T) \wedge \phi = 0. \tag{1.7}$$

Then by Lemma 1.1, there is a unique current in $\mathcal{Z}$ satisfying (1.6).

□

Property 1.3.

Let $\mathcal{X}$ a connected C^∞ manifold of dimension m. Assume it is equipped with a de Rham data. For Lebesgue currents T_1, T_2, the intersection $[T_1 \wedge T_2] \in C(\mathcal{X})$ defined in [6] satisfies:

(1) (Supportivity)

$$supp([T_1 \wedge T_2]) \subset supp(T_1) \cap supp(T_2). \tag{1.8}$$

(2) (Closedness) The intersection current $[T_1 \wedge T_2]$ is closed if T_1, T_2 are.

(3) (Cohomologicity) We use $\langle T \rangle$ to denote the cohomology class represented by a current $T \in \mathscr{L}(\mathcal{X})$. If T_1, T_2 are closed, then in de Rham cohomology we have

$$\langle T_1 \rangle \cup \langle T_2 \rangle = \langle [T_1 \wedge T_2] \rangle. \tag{1.9}$$

Hence if the cohomology $\langle T_1 \rangle, \langle T_2 \rangle$ are integral, so is $\langle [T_1 \wedge T_2] \rangle$.

(4) (Leibniz rule) If dT_1, dT_2 are Lebesgue and $deg(T_1) = p$, then differential of currents follows Leibniz rule,

$$d[T_1 \wedge T_2] = [dT_1 \wedge T_2] + (-1)^p [T_1 \wedge dT_2]. \tag{1.10}$$

(5) (Commutativity) Let $deg(T_1) = p, deg(T_2) = q$. Then

$$[T_1 \wedge T_2] = (-1)^{pq} [T_2 \wedge T_1]. \tag{1.11}$$

Proof. (1) It is Proposition 4.4, [6].

(2) Let ϕ be a test form. By the definition

$$\begin{aligned} &\int_{b[T_1 \wedge T_2]} \phi \\ &= \lim_{\epsilon \to 0} \int_{T_1} R_\epsilon T_2 \wedge d\phi \\ &= \pm \int_{T_1} dR_\epsilon T_2 \wedge \phi \end{aligned} \tag{1.12}$$

According to the homotopy (3.1), [6]

$$bR_\epsilon T_2 - bT_2 = bbA_\epsilon T_2 - bA_\epsilon bT_2 \tag{1.13}$$

Because T_2 is closed,

$$bR_\epsilon T_2 = 0.$$

So $[T_1 \wedge T_2]$ is closed.

(3) Let ϕ be a closed C^∞ form of degree $deg(T_1)+deg(T_2)$, and has a compact support. Denote the cohomology class by $\langle \cdot \rangle$. The intersection number,

$$\Big(\langle [T_1 \wedge T_2] \rangle \Big) \cup \langle \phi \rangle \tag{1.14}$$

is a well-defined real number that equals to

$$\lim_{\epsilon \to 0} \int_{T_1} R_\epsilon (T_2) \wedge \phi. \tag{1.15}$$

By the definition in §20, [1], the integral (1.14) is de Rham's notion

$$\Big([T_1 \wedge \phi] \wedge T_2 \Big) [1].$$

which is the intersection number

$$\Big(\langle T_1\rangle \cup \langle T_2\rangle\Big) \cup \langle \phi\rangle. \tag{1.16}$$

By the duality in Theorem 17, [1], the formulas (1.13) and (1.15) yield

$$\langle T_1\rangle \cup \langle T_2\rangle = \langle [T_1\wedge T_2]\rangle. \tag{1.17}$$

(4) (Leibniz Rule) Let $\phi \in \mathscr{D}(\mathcal{X})$ be a test form. Let

$$deg(T_1) = p, deg(T_2) = q.$$

Then

$$\begin{aligned}
& b[T_1\wedge T_2](\phi) \\
& = \lim_{\epsilon\to 0}\int_{T_1} R_\epsilon T_2 \wedge d\phi \\
& \text{(Leibniz Rule for } C^\infty \text{ forms)} \\
& = \lim_{\epsilon\to 0}\int_{T_1}\Big((-1)^q d(R_\epsilon T_2\wedge\phi) + (-1)^{q+1} dR_\epsilon T_2\wedge\phi\Big) \\
& = \lim_{\epsilon\to 0}\int_{(-1)^q bT_1} R_\epsilon T_2\wedge\phi + \lim_{\epsilon\to 0}\int_{(-1)^{q+1}T_1} dR_\epsilon T_2\wedge\phi \\
& \text{(} bT_1, bT_2 \text{ are Lebesgue)} \\
& = \int_{(-1)^q[bT_1\wedge T_2]}\phi + \int_{(-1)^{q+1}[T_1\wedge dT_2]}\phi
\end{aligned}$$

Hence

$$b[T_1\wedge T_2] = (-1)^q[bT_1\wedge T_2] + (-1)^{q+1}[T_1\wedge dT_2]. \tag{1.18}$$

After change the sign, we found (1.17) is the same as (1.10).

(5) G. de Rham in [1] defined two maps

$$\begin{array}{ccc}
A^r(\mathcal{X}\times\mathcal{Y}) & \xrightarrow{\mathcal{A}^*} & \sum_{i+j=r}\Gamma\Big(A^j(\mathcal{Y})\otimes\wedge^i T^*(\mathcal{X})\Big) \\
\Big\downarrow \mathcal{A}_S^* & & \\
\sum_{i+j=r}\Gamma\Big(A^i(\mathcal{X})\otimes\wedge^j T^*(\mathcal{Y})\Big) & &
\end{array}$$

where $A^\bullet(\cdot)$ denotes the space of C^∞ forms and $\Gamma(E \otimes (\wedge^\bullet T^*(-))$ denotes the space of C^∞ forms with the value in vector space E. Both images are called double forms that are in the isomorphic spaces

$$\Gamma\Big(A^j(\mathcal{Y}) \otimes \wedge^i T^*(\mathcal{X})\Big) \simeq \Gamma\Big(A^i(\mathcal{X}) \otimes \wedge^j T^*(\mathcal{Y})\Big).$$

He stated (p51, [1])

$$\mathcal{A}^*(\phi) - (-1)^{pq}\mathcal{A}_S^*(\phi) = 0$$

if the test form ϕ has the pure degree p in $\mathcal{X}$ and pure degree q in $\mathcal{Y}$.

Recall $\varrho_\epsilon(\mathbf{x}, \mathbf{y})$ is the kernel of R_ϵ, a C^∞ form on $\mathcal{X} \times \mathcal{X}$. The explicit formula is

$$R_\epsilon = \eta \circ \mathcal{A}^*(\varrho_\epsilon(\mathbf{x}, \mathbf{y}))|_y$$

where $\eta = \pm$ is the sign operator dependent of orientations and degrees, and left hand side is the double form evaluated in $\mathbf{y}$ (i.e. with the order 1)$\mathbf{y}$, 2)$\mathbf{x}$).

Then we evaluate the currents in weak limits for above test form ϕ,

$$\begin{aligned}
&\lim_{\epsilon \to 0}\Big([T_1 \wedge R_\epsilon^{\mathcal{X}}(T_2)] - (-1)^{pq}[T_2 \wedge R_\epsilon^{\mathcal{X}}(T_1)]\Big)(\phi) \\
&= \lim_{\epsilon \to 0}\left(\int_{(\mathbf{x},\mathbf{y}) \in T_1 \times T_2} \eta \circ \mathcal{A}^*(\varrho_\epsilon(\mathbf{x}, \mathbf{y})) \wedge \phi - \eta \circ (-1)^{pq}\mathcal{A}_S^*((\varrho_\epsilon(\mathbf{x}, \mathbf{y})) \wedge \phi\right) \\
&\text{(By de Rham's remark above for the order of his double form evaluation)} \\
&= \lim_{\epsilon \to 0}\left(\int_{T_1 \times T_2} 0 \wedge \phi\right) \\
&= 0.
\end{aligned} \tag{1.19}$$

So

$$[T_1 \wedge T_2] - (-1)^{pq}[T_2 \wedge T_1].$$

□

1.2 Advanced properties

Definition 1.4.

Let $\mathcal{U}_1$, $\mathcal{U}_2$ be the de Rham data for the manifolds $\mathcal{X}_1, \mathcal{X}_2$ respectively. We define the product de Rham data on the product $\mathcal{X}_1 \times \mathcal{X}_2$ by taking the Cartesian product of given de Rham data in the following way: if U_i, V_j are the de Rham coverings, the de Rham covering in $X_1 \times X_2$ is (U_i, V_j) with a fixed order of the pair (i, j). We denote the product de Rham data by the same notation $R_\epsilon^{\mathcal{X}_1 \times \mathcal{X}_2}$.

Proposition 1.5. *(Projection formula) Let $\mathcal{X}_1, \mathcal{X}_2$ be two manifolds endowed with de Rham data, $\mathcal{X}_1 \times \mathcal{X}_2$ be endowed with the product de Rham data. Let $P_i : \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{X}_i$ be the projections for $i = 1, 2$ respectively, $\sigma \in C(\mathcal{X}_1)$, and $T \in C(\mathcal{X}_1 \times \mathcal{X}_2)$. Then*

(1)

$$R_\epsilon^{\mathcal{X}_1 \times \mathcal{X}_2}(\sigma \times \mathcal{X}_2) = (P_1)^*(R_\epsilon^{\mathcal{X}_1}(\sigma)) \tag{1.20}$$

where $\sigma \times \mathcal{X}_2$ is the product current–the single current of the tensor product of the currents.

(2) Let $\mathcal{X}_2$ be compact. Then

$$[(P_1)_*(T) \wedge \sigma] = (P_1)_*[T \wedge (\sigma \times \mathcal{X}_2)]. \tag{1.21}$$

where the left hand side is the intersection in $\mathcal{X}_1$, the right hand side is the intersection in $\mathcal{X}_1 \times \mathcal{X}_2$.

(3) If $\mathcal{X}_1 = \mathcal{X}_2 = \mathcal{X}$, then for $\sigma \in \mathscr{L}(X)$ and product de Rham data

$$(P_2)_* \Big[\Delta_{\mathcal{X}} \wedge (\sigma \times \mathcal{X}) \Big] = \sigma \tag{1.22}$$

where $\Delta_{\mathcal{X}}$ is the diagonal of $\mathcal{X}$.

Proof. (1). Assume $\mathcal{X}_1, \mathcal{X}_2$ are endowed with de Rham data, $\mathcal{U}_1, \mathcal{U}_2$ respectively. For the regularization, let's give a product de Rham data to $\mathcal{X}_1 \times \mathcal{X}_2$. We claim for any $\sigma \in C(\mathcal{X})$,

Claim 1.6. *as currents*

$$R_\epsilon^{\mathcal{X}_1 \times \mathcal{X}_2}(\sigma \times \mathcal{X}_2) = R_\epsilon^{\mathcal{X}_1}(\sigma) \times \mathcal{X}_2 \tag{1.23}$$

on $\mathcal{X}_1 \times \mathcal{X}_2 - \partial$.

where ∂ is the union of the boundary of the unit balls in de Rham data (see [6]).

Proof of the claim. Let $B_1 \subset \mathcal{X}_1, B_2 \subset \mathcal{X}_2$ be two unit balls in the de Rham data $\mathcal{U}_1, \mathcal{U}_2$ for $\mathcal{X}_1, \mathcal{X}_2$ respectively. Using the data from B_1, B_2, we construct the local smoothing operators for $B_1 \times B_2$ and B_1. We denote them by $R_\epsilon^{B_1 \times B_2}$ and $R_\epsilon^{B_1}$. Then the direct expression shows

$$R_\epsilon^{B_1 \times B_2}(\sigma|_{B_1} \times B_2) = R_\epsilon^{B_1}(\sigma|_{B_1}) \times B_2$$

as currents on $B_1 \times B_2$. Taking the composition for the de Rham's smoothing operator, both sides stay in the similar type. We obtain

$$R_\epsilon^{\mathcal{X}_1 \times \mathcal{X}_2}(\sigma \times \mathcal{X}_2) = R_\epsilon^{\mathcal{X}_1}(\sigma) \times \mathcal{X}_2. \tag{1.24}$$

Then we obtain that (1.21) holds on the

$$\mathcal{X}_1 \times \mathcal{X}_2 - \partial. \qquad \square$$

Now since both sides of (1.21) are C^∞, by the continuity, (1.21) is extended to the closure $\mathcal{X}_1 \times \mathcal{X}_2$. This completes the proof of part (1).

(2). Since $\mathcal{X}_2$ is compact, P_1 is proper. Then the pushforward $(P_1)_*$ of currents is well-defined. Let ϕ be a test form on $\mathcal{X}_1$. We use the product de Rham data on $\mathcal{X}_1 \times \mathcal{X}_2$ to find

$$\int_{[(P_1)_*(T)\wedge\sigma]} \phi = \lim_{\epsilon\to 0} \int_{(P_1)_*T} R_\epsilon^{\mathcal{X}_1}(\sigma) \wedge \phi$$
$$= \lim_{\epsilon\to 0} \int_T P_1^*(R_\epsilon^{\mathcal{X}_1}(\sigma) \wedge \phi)$$
$$= \lim_{\epsilon\to 0} \int_T P_1^*(R_\epsilon^{\mathcal{X}_1}(\sigma)) \wedge P_1^*(\phi)$$

(Use part (1))

$$= \lim_{\epsilon\to 0} \int_T R_\epsilon^{\mathcal{X}_1\times\mathcal{X}_2}(\sigma \times \mathcal{X}_2) \wedge P_1^*(\phi)$$
$$= \int_{(P_1)_*[T\wedge(\sigma\times\mathcal{X}_2)]} \phi$$

This completes the proof.

(3) For (1.20), we let $\phi \in \mathscr{D}(\mathcal{X})$. Then

$$\int_{(P_2)_*[\Delta_{\mathcal{X}}\wedge(\sigma\times\mathcal{X})]} \phi$$
$$= \int_{[\Delta_{\mathcal{X}}\wedge(\sigma\times\mathcal{X})]} (P_2)^*(\phi)$$
$$= \lim_{\epsilon\to 0} \int_{\Delta_{\mathcal{X}}} R_\epsilon^{\mathcal{X}\times\mathcal{X}}(\sigma \times \mathcal{X}) \wedge (P_2)^*(\phi)$$

(Use projection formula, (1.19))

$$= \lim_{\epsilon\to 0} \int_{\Delta_{\mathcal{X}}} (P_1)^*(R_\epsilon^{\mathcal{X}}(\sigma)) \wedge (P_2)^*(\phi)$$

(Identify $\mathcal{X} \simeq \Delta$)

$$= \lim_{\epsilon\to 0} \int_{\mathcal{X}} R_\epsilon^{\mathcal{X}}(\sigma) \wedge \phi$$
$$= \int_\sigma \phi.$$

□

Delign gave an example showing that the associativity of the current's intersection does not hold ([2]). However, we show another type of associativity still holds.

Proposition 1.7. *(Conditional associativity)*
Let $i : \mathcal{Z} \to \mathcal{X}$ be the embedding of manifolds. There exist de Rham data $\mathcal{U}_{\mathcal{Z}}, \mathcal{U}_{\mathcal{X}}$ on $\mathcal{Z}, \mathcal{X}$ respectively such that for $\mathcal{W} \in \mathscr{L}(\mathcal{Z})$, and $\sigma \in \mathscr{L}(\mathcal{X})$,

$$i_*([\mathcal{W} \wedge_{\mathcal{Z}} [\mathcal{Z} \wedge_{\mathcal{X}} \sigma]_{\mathcal{Z}}]) = [i_*\mathcal{W} \wedge_{\mathcal{X}} \sigma] \tag{1.25}$$

where the notation for $[\mathcal{Z} \wedge \sigma]_{\mathcal{Z}}$ is defined in Proposition 2.5, and the subscript under $\wedge$ denotes the ambient space of the intersection.

Proof. We may assume $\mathcal{Z}$ is compact. Let $j : E \to \mathcal{Z}$ be a tubular neighborhood of $\mathcal{Z}$ in $\mathcal{X}$. Thus E is diffeomorphic to a vector bundle of rank r. We denoted the bundle also by E. Let $i : \mathcal{Z} \to E$ is the 0-section embedding. Let $\mathcal{U}$ be a de Rham data for $\mathcal{Z}$ such that each de Rham chart U_i lies in the trivialization. So

$$j^{-1}(U_i) \simeq U_i \times \mathbb{R}^r. \tag{1.26}$$

Let $j^{-1}(U_i), i$ be the de Rham covering for E. On each $j^{-1}(U_i)$, we use the product de Rham data for $U_i \times \mathbb{R}^r$. Then as for the construction of de Rham's smoothing operator [6], we glue them to obtain the de Rham data for E, denoted by $\mathcal{U}_E$. At last we extend it arbitrarily to the whole manifold $\mathcal{X}$ to have a de Rham data $\mathcal{U}_{\mathcal{X}}$. Notice $\mathcal{W}$ is supported around $\mathcal{Z}$. Hence the intersection occur in E. We may continue with $\mathcal{X} = E$. Then it suffices to work in one chart

$$j^{-1}(U_i) \simeq U_i \times \mathbb{R}^r$$

that is equipped with the product de Rham data. Let the projection of E to U_i be π_1, and to $\mathbb{R}^r$ π_2. Then

$$[\sigma \wedge_{\mathcal{X}} \mathcal{Z}] = \lim_{\epsilon \to 0} [\sigma \wedge_{\mathcal{X}} R_\epsilon^{\mathcal{X}}(\mathcal{Z})] \tag{1.27}$$

where the limit is the weak limit for currents in $\mathcal{X}$. We continue to have

$$\Big[[\sigma \wedge_{\mathcal{X}} \mathcal{Z}]_{\mathcal{Z}} \wedge \mathcal{W}\Big] = \lim_{\epsilon \to 0} \lim_{\epsilon' \to 0} \Big[[\sigma \wedge_{\mathcal{X}} (\pi_2)^*(R_\epsilon^{\mathcal{X}})]_{\mathcal{Z}} \wedge_{\mathcal{Z}} R_{\epsilon'}^{\mathcal{Z}}(\mathcal{W})\Big] \tag{1.28}$$

Since two parameters ϵ, ϵ' are located in two independent differential forms, the order of the iterated limit can be exchanged. So we have

$$\begin{aligned} & i_*\Big[[\sigma \wedge_{\mathcal{X}} \mathcal{Z}]_{\mathcal{Z}} \wedge_{\mathcal{Z}} \mathcal{W}\Big] \\ & = i_*\Big(\lim_{\epsilon' \to 0} \lim_{\epsilon \to 0} \Big[[\sigma \wedge_{\mathcal{X}} (\pi_2)^*(R_\epsilon^{\mathcal{X}}(\mathcal{Z}))]_{\mathcal{Z}} \wedge_{\mathcal{Z}} R_{\epsilon'}^{\mathcal{Z}}(\mathcal{W})\Big]\Big) \\ & (\text{ Note } \lim_{\epsilon' \to 0} (\pi_2)^*(R_\epsilon^{\mathcal{X}}(\mathcal{Z})) \wedge_{\mathcal{X}} (\pi_2)^*(R_{\epsilon'}^{\mathcal{Z}}(\mathcal{W})) = R_\epsilon^{\mathcal{X}}(\mathcal{W})) \\ & = \lim_{\epsilon \to 0} [\sigma \wedge_{\mathcal{X}} R_\epsilon^{\mathcal{X}}(\mathcal{W}))] \\ & = [\sigma \wedge_{\mathcal{X}} \mathcal{W}]. \end{aligned} \tag{1.29}$$

By the commutativity of the intersection,

$$i_*\Big[\mathcal{W} \wedge_{\mathcal{Z}} [\mathcal{Z} \wedge_{\mathcal{X}} \sigma]_{\mathcal{Z}}\Big] = [\mathcal{W} \wedge_{\mathcal{X}} \sigma].$$

We complete the proof. □

Definition 1.8. *Any pair of de Rham data such as $\mathcal{U}_{\mathcal{Z}}, \mathcal{U}_{\mathcal{X}}$ satisfying Proposition 2.12 will be called the associative de Rham data.*

We further establish formulas in cohomology. Let

$$\mathscr{L}_C(\mathcal{X}) \subset \mathscr{L}(\mathcal{X})$$

be the subgroup of closed currents.

Proposition 1.9.
Let $\mathcal{X}, \mathcal{X} \times \mathcal{X}$ be endowed with de Rham data. Let $T_1, T_2, T_3 \in \mathscr{L}(\mathcal{X})$.

(1) Reduction to the diagonal
There exists a homologically trivial current α_1 such that

$$[T_1 \wedge T_2] = (-1)^m (P_2)_*[\Delta \wedge (T_1 \times T_2)] + \alpha_1 \tag{1.30}$$

where $P_2 : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$(2nd copy) is the projection, Δ is the diagonal.

(2) Commutativity
There exists a homologically trivial current α_2 such that

$$[T_1 \wedge T_2] = (-1)^{dim(T_1)dim(T_2)}[T_2 \wedge T_1] + \alpha_2. \tag{1.31}$$

(3) Associativity
There exists a homologically trivial current α_3 such that

$$\Big[T_1 \wedge [T_2 \wedge T_3]\Big] = \Big[[T_1 \wedge T_2] \wedge T_3]\Big] + \alpha_3. \tag{1.32}$$

Let $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ be the inclusion of a submanifold endowed with a de Rham data. Let $\mathcal{W} \in \mathscr{L}(\mathcal{Z})$, and $\sigma \in \mathscr{L}(\mathcal{X})$. Then there exists a homologically trivial current α_4 such that

$$i_*([\mathcal{W} \wedge_{\mathcal{Z}} [\mathcal{Z} \wedge_{\mathcal{X}} \sigma]_{\mathcal{Z}}]) = [i_*\mathcal{W} \wedge_{\mathcal{X}} \sigma] + \alpha_4 \tag{1.33}$$

where the subscript in $\wedge_\bullet$ denotes the ambient space in intersection.

Proof. Notice the cohomologicity

$$\langle T_1 \rangle \cup \langle T_2 \rangle = \langle [T_1 \wedge T_2] \rangle. \tag{1.34}$$

holds, and the proper pushforward of currents is compatible with the pullback of cohomology. Thus all formulas in propositions follow from their corresponding versions in cohomology.

□

2 Dependence of de Rham data

The intersection of currents depends on extrinsic de Rham data. However, in classical cases the intersection does not depend on any extrinsic data. In this section, we'll show that two views have no contradiction since when the currents have certain geometric structures, the dependence vanishes. So the dependence plays a profound role in the intrinsic definition of intersection.

2.1 Real case

It is well-known that on a manifold, if two submanifolds meet transversally at another submanifold, then the intersection should be defined to be the intersectional manifold. The more useful version is its extension to algebraic geometry. The following proposition says that the transversal intersection is a particular case where the dependence of de Rham data disappears due to the special geometric position.

Proposition 2.1. *Let $\mathcal{X}$ be a manifold of dimension m. If T_1, T_2 are cells of real dimension p, q with $p+q \geq m$, and the intersection $T_1 \cap T_2$ is transversal at a connected, manifold V of dimension $p+q-m$. Then $[T_1 \wedge T_2]$ is independent of de Rham data. Furthermore it is the current of integration over V.*

Proof. Let's set up the coordinates for the cells. Let $\mathcal{X} = \mathbb{R}^m$ have linear basis $\mathbf{e}_1, \cdots, \mathbf{e}_m$ and coordinates $x_1, \cdots, x_m$. Set up the subspaces,

$$\begin{aligned}
\mathbb{R}^p &= span(\mathbf{e}_1, \cdots, \mathbf{e}_p) \\
\mathbb{R}^q &= span(\mathbf{e}_{m-q+1}, \cdots, \mathbf{e}_m) \\
\mathbb{R}^{p+q-m} &= span(\mathbf{e}_{m-q+1}, \cdots, \mathbf{e}_p)
\end{aligned}$$

Let $T_1=\Delta^p\subset\mathbb{R}^p$ be the polyhedron defined by

$$\{\sum_{i=1}^{p}|x_i|<1\} \tag{2.1}$$

Similarly $T_2=\Delta^q$ is defined by

$$\{\sum_{i=m-q+1}^{m}|x_i|<1\}, \tag{2.2}$$

$V=\Delta^p\cap\Delta^q$ is defined by

$$\{\sum_{i=m-q+1}^{p}|x_i|<1\}. \tag{2.3}$$

Let $\pi_{p+q-m}:\mathbb{R}^m\to\mathbb{R}^{p+q-m}$ be the projection. The proof has two steps.

1st step: Notice $[T_1\wedge T_2]$ has a compact support, hence $[T_1\wedge T_2]$ is also evaluated at the forms in $C^\infty(\mathbb{R}^m)$ without a compact support. Let $\phi\in\mathscr{D}(\mathbb{R}^m)$. Notice by the definition $[T_1\wedge T_2]$ is $i_*([T_1\wedge T_2]_{\mathbb{R}^{p+q-m}})$ for some current

$$[T_1\wedge T_2]_{\mathbb{R}^{p+q-m}} \tag{2.4}$$

in $\mathbb{R}^{p+q-m}$, where $i:\mathbb{R}^{p+q-m}\hookrightarrow\mathbb{R}^m$ is the inclusion map. We denote $i^*(\phi)$ by ϕ_0. Then we obtain that

$$\begin{aligned}\int_{[T_1\wedge T_2]}\phi&=\int_{[T_1\wedge T_2]_{\mathbb{R}^{p+q-m}}}\phi_0\\&=\int_{[T_1\wedge T_2]}\pi^*_{p+q-m}(\phi_0)\end{aligned} \tag{2.5}$$

Recall that the C^∞ form $\pi^*_{p+q-m}(\phi_0)$ is not compactly supported, and is called a local constant slicing in Definition 3.5, [6], therefore a closed form. So

$$d(\pi^*_{p+q-m}(\phi_0))=0.$$

Now we apply the homotopy formula (3.1), [6]. It suffices to work with $\phi\in\mathscr{D}(\mathbb{R}^m)$ such that

$$supp(\pi^*_{p+q-m}(\phi_0))\cap\big(\partial(T_1)\cup\partial(T_2)\big)=\varnothing.$$

For arbitrary de Rham's regularization R'_ϵ, A'_ϵ with fixed sufficiently small real numbers ϵ_1,ϵ_2, we apply the homotopy formula (3.1), [5] to have

$$\int_{[T_1\wedge T_2]}\pi^*_{p+q-m}(\phi_0) \tag{2.6}$$

$$=-\int_{[(bA'_{\epsilon_1}T_1+A'_{\epsilon_1}bT_1)\wedge(bA'_{\epsilon_2}T_2+A'_{\epsilon_2}bT_2)]}\pi^*_{p+q-m}(\phi_0) \tag{2.7}$$

$$+\int_{[R'_{\epsilon_1}T_1\wedge R'_{\epsilon_2}T_2]}\pi^*_{p+q-m}(\phi_0) \tag{2.8}$$

Now we calculate the first integral (2.7)

$$\begin{aligned}
&\int_{[(bA'_{\epsilon_1}T_1+A'_{\epsilon_1}bT_1)\wedge(bA'_{\epsilon_2}T_2+A'_{\epsilon_2}bT_2)]}\pi^*_{p+q-m}(\phi_0)\\
&(\text{because } supp(bT_i)\cap supp(\pi^*_{p+q-m}(\phi_0))=\varnothing.)\\
&=\pm\lim_{\epsilon\to 0}\int_{[A_{\epsilon'_1}T_1\wedge bA'_{\epsilon_2}T_2]} d(\pi^*_{p+q-m}(\phi_0))\\
&=0
\end{aligned}$$

This shows that

$$\int_{[T_1\wedge T_2]}\pi^*_{p+q-m}(\phi_0)=\int_{[R'_{\epsilon_1}T_1\wedge R'_{\epsilon_2}T_2]}\pi^*_{p+q-m}(\phi_0). \tag{2.9}$$

We observe that the right hand side of (2.9) does not involve the de Rham's smoothing operator R_ϵ, thus the current $[T_1\wedge T_2]$ is independent of the choice of de Rham data $\mathcal{U}$.

2_nd_ step: To calculate the intersection $[T_1\wedge T_2]$. By the 1st step, we can choose a particular de Rham's data $\mathcal{U}$ that has one chart $x_1,\cdots,x_m$ for $\mathbb{R}^m$. Also we choose a C^∞ convolution function $f(\mathbf{x})$ supported in a neighborhood of a unit ball B satisfying

$$\int_{\mathbb{R}^m} f(\mathbf{x})d\mu=1. \tag{2.10}$$

where $d\mu=dx_1\wedge\cdots\wedge dx_m$. Let $\vartheta_\epsilon(x)=f(\frac{\mathbf{x}}{\epsilon})d\frac{\mu}{\epsilon}$.

Let

$$\begin{array}{rcl}\kappa:\mathbb{R}^m\times\mathbb{R}^m & \to & \mathbb{R}^m\\ (\mathbf{x},\mathbf{y}) & \to & \mathbf{x}-\mathbf{y},\end{array} \tag{2.11}$$

Denote the coordinates $(x_1,\cdots,x_{m-q})$ by $\mathbf{x}_1$, $(x_{m-q+1},\cdots,x_p)$ by $\mathbf{x}_2$ and $x_{i+1},\cdots,x_m$ by $\mathbf{x}_3$. Similarly for the second copy of $\mathbb{R}^m$ in (2.11), the corresponding coordinates are denoted by $\mathbf{y}_1,\mathbf{y}_2,\mathbf{y}_3$ respectively.

Let

$$g(\frac{\mathbf{x}_1}{\epsilon},\frac{\mathbf{x}_2}{\epsilon},\frac{\mathbf{x}_3}{\epsilon},\frac{\mathbf{y}_1}{\epsilon},\frac{\mathbf{y}_2}{\epsilon},\frac{\mathbf{y}_3}{\epsilon})=\kappa^*(\vartheta_\epsilon). \tag{2.12}$$

Let $\phi\in\mathscr{D}(\mathbb{R}^m)$ be a test form. Then we calculate the current

$$\begin{aligned}
&\int_{[T_1\wedge T_2]}\phi=\lim_{\epsilon\to 0}\int_{T_1}R_\epsilon(T_2)\wedge\phi\\
&=\lim_{\epsilon\to 0}\int_{T_1}\int_{(\mathbf{y}_2,\mathbf{y}_3)\in T_2} g(\frac{\mathbf{x}_1}{\epsilon},\frac{\mathbf{x}_2}{\epsilon},0,0,\frac{\mathbf{y}_2}{\epsilon},\frac{\mathbf{y}_3}{\epsilon})\wedge\phi(\epsilon\frac{\mathbf{x}_1}{\epsilon},\mathbf{x}_2,0)
\end{aligned} \tag{2.13}$$

where $\phi(\epsilon\frac{\mathbf{x}_1}{\epsilon},\mathbf{x}_2,0)$ is a test form, i.e. C^∞ form on T_1 with a compact support.

Now applying the fibre integral to that over T_1, we obtain

$$\int_{[T_1\wedge T_2]}\phi$$
$$= \lim_{\epsilon\to 0}\int_{\mathbf{x}_2\in\mathbb{R}^{i+j-m}}\int_{\mathbf{x}_1\in\mathbb{R}^{m-j}}\int_{(\mathbf{y}_2,\mathbf{y}_3)\in\mathbb{R}^j} g(\frac{\mathbf{x}_1}{\epsilon},\frac{\mathbf{x}_2}{\epsilon},0,0,\frac{\mathbf{y}_2}{\epsilon},\frac{\mathbf{y}_3}{\epsilon})\wedge\phi(\epsilon\frac{\mathbf{x}_1}{\epsilon},\mathbf{x}_2,0) \tag{2.14}$$

Then we make a change of variables,

$$\begin{aligned}\tfrac{\mathbf{x}_1}{\epsilon}&\to\mathbf{x}_1,\\ \tfrac{\mathbf{y}_2}{\epsilon}&\to\mathbf{y}_2\\ \tfrac{\mathbf{y}_3}{\epsilon}&\to\mathbf{y}_3.\end{aligned} \tag{2.15}$$

Then

$$\int_{[T_1\wedge T_2]}\phi$$
$$= \pm\lim_{\epsilon\to 0}\int_{\mathbf{x}_2\in\mathbb{R}^{i+j-m}}\int_{\mathbf{x}_1\in\mathbb{R}^{m-j}}\int_{(\mathbf{y}_2,\mathbf{y}_3)\in\mathbb{R}^j} g(\mathbf{x}_1,\frac{\mathbf{x}_2}{\epsilon},0,0,\mathbf{y}_2,\mathbf{y}_3)\wedge\phi(\epsilon\mathbf{x}_1,\mathbf{x}_2,0) \tag{2.16}$$

Then we notice for each fixed $\mathbf{x}_2$, the fibre integral

$$\int_{\mathbf{y}_2,\mathbf{y}_3\in\mathbb{R}^j,\mathbf{x}_1\in\mathbb{R}^{m-j}} g(\mathbf{x}_1,\frac{\mathbf{x}_2}{\epsilon},0,0,\mathbf{y}_2,\mathbf{y}_3) \tag{2.17}$$

by the formula (2.10), is 1. Therefore we obtain that

$$[T_1\wedge T_2](\phi)=\int_{\mathbb{R}^{i+j-m}}\phi(0,\mathbf{x}_2,0) \tag{2.18}$$

Thus

$$[T_1\wedge T_2](\phi)=\textstyle\int_V \phi|_V. \tag{2.19}$$

where $\phi|_V$ is the restriction $\phi(0,\mathbf{x}_2,0)$ of ϕ to V. We complete the proof.

□

For non transversal intersection, there is no clear notion of geometric position. So the dependence varies from case to case. The following examples are all based on singular chains.

Example 2.2. *(real excess intersection)*
Let $\mathcal{X}=\mathbb{R}^2$, and be equipped with the de Rham data consisting of single chart $\mathbb{R}^2$ with the convolution function f satisfying

$$\int_{\mathbb{R}^2} f(x_1,x_2)dx_1\wedge dx_2 = 1 \tag{2.20}$$

where x_1, x_2 are Euclidean coordinates of $\mathbb{R}^2$. Let $T_1 = T_2$ be the current of integration over the finite piece of the parabola

$$x_1 = x_2^2 \tag{2.21}$$

containing the origin $\mathbf{0}$. Since T_1, T_2 are singular chains, $[T_1 \wedge T_2]$ exists. Let $\phi(x)$ be a test function with a compact support. Denote the second copy of $\mathbb{R}^2$ for the de Rham's regularization by y_1, y_2. Then we calculate

$$\begin{array}{c} \int_{[T_1\wedge T_2]} \phi \\ \| \\ \lim_{\epsilon\to 0} \frac{1}{\epsilon^2} \int_{x\in T_1} \int_{y\in T_2} f(\frac{x_1-y_1}{\epsilon}, \frac{x_2-y_2}{\epsilon})\phi(x_1,x_2)(dx_1-dy_1)\wedge(dx_2-dy_2) \end{array} \tag{2.22}$$

substitute $x_1 = x_2^2, y_1 = y_2^2$ for T_1, T_2, we obtain that

$$\begin{array}{c} \int_{[T_1\wedge T_2]} \phi \\ \| \\ \lim_{\epsilon\to 0} \frac{2}{\epsilon^2} \int_{x_2\in\mathbb{R}} \int_{y_2\in\mathbb{R}} f(\frac{(x_2-y_2)(x_2+y_2)}{\epsilon}, \frac{x_2-y_2}{\epsilon})\phi(x_1,x_2)(x_2-y_2)dy_2\wedge dx_2. \end{array} \tag{2.23}$$

Next we make a change of the variables

$$\begin{cases} u = \frac{(x_2-y_2)}{\epsilon} \\ v = x_2 + y_2. \end{cases} \tag{2.24}$$

Then

$$\begin{array}{c} [T_1\wedge T_2](\phi) \\ \| \\ \lim_{\epsilon\to 0} \int_{u\in\mathbb{R}} \int_{v\in\mathbb{R}} uf(uv,u)\phi((\frac{\epsilon u+v}{2})^2, \frac{\epsilon u+v}{2})dv\wedge du \\ \| \\ \int_{(u,v)\in\mathbb{R}^2} uf(uv,u)\phi((\frac{v}{2})^2, \frac{v}{2})dv\wedge du. \end{array} \tag{2.25}$$

Then the functional

$$\phi \to \int_{(u,v)\in\mathbb{R}^2} uf(uv,u)\phi((\frac{v}{2})^2, \frac{v}{2})dv\wedge du \tag{2.26}$$

defines a current supported on T_1. So the intersection current

$$[T_1\wedge T_2]$$

(which is (2.26)) is supported on T_1, depending on the convolution function f.

Example 2.3. *(real proper intersection)*
Let $\mathcal{X} = \mathbb{R}^2$ be equipped with the de Rham data consisting of single chart $\mathbb{R}^2$ with the convolution function f satisfying

$$\int_{\mathbb{R}^2} f(x_1, x_2) dx_1 \wedge dx_2 = 1 \tag{2.27}$$

where x_1, x_2 are Euclidean coordinates of $\mathbb{R}^2$.

Case 1: Let T_1 be a line through the origin $\mathbf{0}$ and T_2 is another line segment through the origin. Then it is known that

$$[T_1 \wedge T_2] = \delta_{\mathbf{0}}$$

if the order matches with the orientation of $\mathbb{R}^2$.

Case 2: Continuing from the setting in case 1, let T_2 be the line $x_1 = 0$. Let T_1 be a piece of parabola

$$x_1 = x_2^2, x_2 \in (-1, 1). \tag{2.28}$$

Let's calculate $[T_1 \wedge T_2]$. Let $\phi(x)$ be a test function supported in a neighborhood of the origin. We denote the second copy of $\mathbb{R}^2$ for de Rhams' regularization by (y_1, y_2). Then

$$\begin{aligned} &\int_{[T_1 \wedge T_2]} \phi \\ &= \lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{x_1 \in T_1} \int_{y_2 \in \mathbb{R}} f(\frac{x_1}{\epsilon}, \frac{x_2}{\epsilon} - \frac{y_2}{\epsilon}) \phi(x_1, x_2) dy_2 \wedge dx_1. \end{aligned} \tag{2.29}$$

Let

$$f_1(x_1) = \int_{y_2 \in \mathbb{R}} f(x_1, -y_2) dy_2.$$

Now we continue (2.29) to have

$$\begin{array}{c} \int_{[T_1 \wedge T_2]} \phi \\ \| \\ \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{(x_1, x_2) \in T_1} f_1(\frac{x_1}{\epsilon}) \phi(x_1, x_2) dx_1 \\ \| \\ \phi(\mathbf{0}) \left(\int_{+\infty}^0 f_1(x_1) dx + \int_0^{+\infty} f_1(x_1) dx_1 \right) = 0, \end{array} \tag{2.30}$$

So

$$[T_1 \wedge T_2] = 0$$

for all convolution function f in the de Rham data. This example shows the formula

$$supp([T_1 \wedge T_2]) = supp(T_1) \cap supp(T_2)$$

does not hold for singular chains.

Case 3: Continuing from the setting in case 2, let T_2 be the line $x_1 = 0$. Let T_1 be a piece of the cubic curve

$$x_1 = x_2^3, x_2 \in (-1, 1). \tag{2.31}$$

The same calculation in case 2 shows if order of T_1, T_2 is concordant with orientation of $\mathbb{R}^2$, then

$$\int_{[T_1 \wedge T_2]} \phi = \phi(\mathbf{0}). \tag{2.32}$$

Hence

$$[T_1 \wedge T_2] = \delta_{\mathbf{0}} \tag{2.33}$$

where $\delta_{\mathbf{0}}$ is the δ-function at the origin. So the intersection is independent choice of convolution function f in de Rham data.

Remark All three cases in Example 2.3 coincide with de Rham's Kronecker index $T_1 \wedge T_2[1]$ which does not depend on the de Rham data.

2.2 Complex case

In complex geometry, intersection of currents coincides with proper intersection where the theory has been explored in great detail through the tool in commutative algebra.

Proposition 2.4. *Let $f : X \to Y$ be a regular map between two smooth projective varieties. Let W be an algebraic cycle of X. We denote the pushfoward of currents and algebraic cycles by the same notation f_*. Then the current $f_*[W]$ is the current of integration over the cycle*

$$f_* W,$$

where $[W]$ stands for the current of integration over the algebraic set.

Proof. Let $W = \sum_i a_i W_i$ where W_i are irreducible subvarieties of the same dimension and a_i are non-zero integers. Let $f_* W_i = b_i S_i$ where b_i is the dimension of field extension of the rational field of S_i to W_i. Let $|W_0|$ be the open sets of the support $|W|$ such that f is smooth. Then correspondingly $f(|W_0|) = \cup_i S_i^0$, where S_i^0 are open sets of S_i. Then using currents, we have

$$f_*[f^{-1}(S_i^0)] = b_i S_i^0. \tag{2.34}$$

Taking the closure and the sum over all i, we obtain that

$$f_*(\sum_i a_i[W_i]) = \sum_i a_i b_i [S_i]. \tag{2.35}$$

Since for algebraic cycles, we have

$$f_*(\sum_i a_i W_i) = \sum_i a_i b_i S_i, \tag{2.36}$$

we complete the proof.

□

Due to the proposition, throughout letter W will denote both currents and algebraic cycles, and f_* denotes the operations on both currents and algebraic cycles.

Theorem 2.5. *Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Let T_1, T_2 be algebraic cycles of dimension p, q in X, To abuse the notations, the currents of integration over them are also denoted by T_1, T_2 respectively. Assume $|T_1| \cap |T_2|$ is proper. Then with an arbitrary de Rham data $\mathcal{U}$ on X, the current $[T_1 \wedge T_2]$ is independent of $\mathcal{U}$, and equals to the current of integration over the algebraic cycle*

$$T_1 \bullet T_2,$$

where $T_1 \bullet T_2$ is the cycle-intersection by Serre's Tor formula. More precisely it is the sum

$$\sum_j m_j W_j.$$

where W_j are all irreducible subvarieties, m_i are intersection multiplicities at W_j if W_j are components with the proper dimension in the intersection scheme.

Proof. It suffices to assume T_1, T_2 are prime cycles (cycles of irreducible subvarieties). Let's fix the cycle T_2. By example 11.4.2, [3], there is an algebraic cycle E_1 rationally equivalent to T_1 such that E_1 meets T_2 transversely (at an open set of each irreducible support). Without losing the generality, let's have a simplified setting as follows. Let $V \subset \mathbf{P}^1 \times X$ be an irreducible subvariety, and $P_2 : V \to X$, $P_1 : V \to \mathbf{P}^1$ are the projections. Let $T_2 \subset X$ be an irreducible subvariety. Assume the cycle of the scheme $P_1^{-1}(1)$ is E_1 and the cycle of the scheme $P_1^{-1}(0)$ is T_1, where $0, 1$ are two points of $\mathbf{P}^1$. Let I be a real curve in $\mathbf{P}^1$ connecting $0, 1$. Next we consider two objects: currents and algebraic cycles. Using the currents, according to Proposition 3.13 (current-homotopy) below in section 3, we have the formula

$$[T_1 \wedge T_2] - [E_1 \wedge T_2] = d\mathcal{W} \tag{2.37}$$

where $\mathcal{W}$ is some current.

Next we consider the algebraic cycles in intersection theory where two rationally equivalent algebraic cycles are homotopic. More precisely, since the intersection $T_1 \cap T_2$ is proper, we have the equation in singular cycles

$$T_1 \bullet T_2 - E_1 \bullet T_2 = dW \tag{2.38}$$

where W is a singular chain in the complex manifold $\mathbf{P}^1 \times X$, $T_1 \bullet T_2, E_1 \bullet T_2$ are singular cycles obtained from the triangulation of the intersectional algebraic cycles, and d is the differential operator on the singular chains (the boundary operator with a sign). We claim

Claim 2.6. *$\mathcal{W}$ is the integration functional over the singular chain W.*

Proof of the claim. We assume T_1, T_2 are irreducible subvarieties. Let D be an irreducible component of the scheme

$$(\mathbf{P}^1 \times T_2) \cap V$$

containing an irreducible component of

$$(\mathbf{P}^1 \times T_2) \cap (\{1\} \times E_1).$$

Since the intersection $E_1 \cap T_2$ is transversal, by the continuity, the intersection $(\mathbf{P}^1 \times T_2) \cap (\{1\} \times E_1)$ at D is genereically transversal. By Proposition 2.1, the current at D, denoted by

$$[(\mathbf{P}^1 \times T_2) \wedge V]|_D \tag{2.39}$$

is the integration over the algebraic subvariety

$$(\mathbf{P}^1 \times T_2) \cap V$$

at D, where $\cdot|_D$ denotes the restriction of a current. Then the restriction current

$$[(I \times X) \wedge V]|_D$$

is an semi-algebraic set which is a singular chain. Taking the union over all components D, we obtain the current $\mathcal{W}$ is the integration over the semi-algebraic set

$$(I \times X) \cap \bigg((\mathbf{P}^1 \times T_2) \cap V\bigg).$$

Let W denote this semi-algebraic set. So $\mathcal{W}$ is the integration over the singular chain W. Linearly extending the assertion to cycles T_1, T_2, we complete the proof of the claim.

By the claim,

$$[T_1 \wedge T_2] - [E_1 \wedge T_2] = T_1 \bullet T_2 - E_1 \bullet T_2 \tag{2.40}$$

where the algebraic cycles at right are regarded as the currents of integration. By Proposition 2.1, since E_1 meets T_2 transversely,

$$[E_1 \wedge T_2] = E_1 \bullet T_2. \tag{2.41}$$

Thus

$$[T_1 \wedge T_2] = T_1 \bullet T_2.$$

We complete the proof.

□

Example 2.7. *Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Let T_1, T_2 be subvarieties of X of codimension p, q. The currents of integration over them are also denoted by T_1, T_2 respectively. Assume $T_1 \cap T_2$ is an excess intersection. Then*

$$[T_1 \wedge T_2] \tag{2.42}$$

in general depends on the de Rham data $\mathcal{U}$.

Let $\mathbf{P}^2$ be a projective space over $\mathbb{C}$ with affine coordinates (z_1, z_2). Let T_1 be the hyperplane $z_2 = 0$, and $T_2 = T_1$. First it is not zero because its reduction to cohomology group is non-zero. Choose two open sets as de Rham's covering: U_1, the finite affine plane, and a small neighborhood U_2 of the infinity $\mathbf{P}^1 \subset \mathbf{P}^2$. Choose real Euclidean coordinates x_1, y_1, x_2, y_2 for U_1 such that

$$z_1 = x_1 + iy_2, z_2 = x_2 + iy_2.$$

Use these open covering and Euclidean coordinates to have a de Rham data for $\mathbf{P}^2$ with a convolution function $h(x_1, x_2, y_1, y_2)$ of the unit ball B in U_1. Then we see in U_1,

$$R^1_\epsilon(T_2) = -\frac{1}{\epsilon^4} \iint_{(x_1', y_1') \in \mathbb{R}^2} h(\frac{x_1 - x_1'}{\epsilon}, \frac{x_2}{\epsilon}, \frac{y_1 - y_1'}{\epsilon}, \frac{y_2}{\epsilon}) dx_1' \wedge dy_1' \wedge dx_2 \wedge dy_2, \tag{2.43}$$

where x_i', y_i' are the Euclidean coordinates for the second factor in the smoothing operator. The composing with another local smoothing operator from U_2 will not change the smooth current $R^1_\epsilon(T_2)$ in B. Thus for a test form ϕ supported in B, the integral

$$\int_{T_1} R^B_\epsilon(T_2) \wedge \phi = \iint_{x_2 = y_2 = 0} (\cdots) dx_2 \wedge dy_2 = 0. \tag{2.44}$$

This shows with this type of de Rham data,

$$[T_1 \wedge T_2] \tag{2.45}$$

is zero on $U \cap T_1$. Hence $[T_1 \cap T_2]$ is a 0-dimensional current supported at the infinity point of T_1. Since the $\infty = \mathbf{P}^1$ is arbitrary, $[T_1 \wedge T_2]$ is supported on an arbitrary set determined by the de Rham data.

Example 2.8.

The following table lists the difference in case of excess intersection.

Table 1: Excess intersection in complex case

Intersection	Cycle	Chow class	Support
algebraic $T_1 \bullet T_2$	not well-defined	well-defined	$\lvert T_1\rvert \cap \lvert T_2\rvert$
current $[T_1 \wedge T_2]$	well-defined	not well-defined	$\lvert T_1\rvert \cap \lvert T_2\rvert$

3 Intersectional operators

We'll define operators for currents based on the intersection of currents.

3.1 Correspondence of a current

Lemma 3.1. *Let $\mathcal{X}, \mathcal{Y}$ be two compact manifolds, and $P_{\mathcal{X}}$ be the projection*

$$\mathcal{X} \times \mathcal{Y} \to \mathcal{X}.$$

Then the image of the projection

$$(P_{\mathcal{X}})_* : C(\mathcal{X} \times \mathcal{Y}) \quad \to \quad \mathscr{D}'(\mathcal{X})$$

lies in $C(\mathcal{X})$.

Proof. Notice there is a coordinates chart of $\mathcal{X} \times \mathcal{Y}$ satisfying that the coordinates planes of $\mathcal{X}$ are also the coordinates planes for $\mathcal{X} \times \mathcal{Y}$. Thus the two conditions of Lebesgue currents for $\mathcal{X}$ are implied by that for $\mathcal{X} \times \mathcal{Y}$.

□

Definition 3.2.

Let $\mathcal{X}, \mathcal{Y}$ be two compact manifolds.

Let

$$F \in C(\mathcal{X} \times \mathcal{Y}) \tag{3.1}$$

be a Lebesgue current. Assume $\mathcal{X} \times \mathcal{Y}$ is equipped with a de Rham data. Let $P_{\mathcal{X}}, P_{\mathcal{Y}}$ be the projections

$$\mathcal{X} \times \mathcal{Y} \to \mathcal{X}, \quad \mathcal{X} \times \mathcal{Y} \to \mathcal{Y}.$$

Define the correspondence $F_{\circledast}(T)$ of currents by

$$\begin{array}{rcc} F_{\circledast} : C(\mathcal{X}) & \to & C(\mathcal{Y}) \\ T & \to & (P_{\mathcal{Y}})_*[F \wedge (T \times \mathcal{Y})]. \end{array} \tag{3.2}$$

Define the transpose

$$F^{\circledast}(T)$$

by

$$\begin{array}{rcl} F^{\circledast}: C(\mathcal{Y}) & \to & C(\mathcal{X}) \\ T & \to & (P_{\mathcal{X}})_*[F \wedge (\mathcal{X} \times T)]. \end{array} \tag{3.3}$$

Proposition 3.3. *Let $\mathcal{X}, \mathcal{Y}$ be compact manifolds. The pull-back and push-forward of currents extents Gillet and Soulé's proper push-forward and smooth pullback on complex manifolds.*

Proof. We verify that Gillet-Soulé's operators coincide with current's correspondence on C^∞ manifolds.

Let

$$f: \mathcal{X} \to \mathcal{Y} \tag{3.4}$$

be a C^∞ map. Let F be its graph. Let T be a Lebesgue current on $\mathcal{X}$. Let ϕ be a C^∞ form on $\mathcal{Y}$. We use a product de Rham data on $\mathcal{X} \times \mathcal{Y}$. Then

$$\begin{aligned} &\int_{F_*(T)} \phi \\ &= \lim_{\epsilon \to 0} \int_F R_\epsilon^{\mathcal{X}\times\mathcal{Y}}(T \times \mathcal{Y}) \wedge (P_{\mathcal{Y}})^*(\phi) \\ &\text{(by Proposition 1.5, the projection formula)} \\ &= \lim_{\epsilon \to 0} \int_F (P_{\mathcal{X}})^* R_\epsilon^{\mathcal{X}}(T) \wedge (P_{\mathcal{Y}})^*(\phi) \\ &= \lim_{\epsilon \to 0} \int_{\mathcal{X}} R_\epsilon^{\mathcal{X}}(T) \wedge f^*(\phi) \\ &= \int_T f^*(\phi). \end{aligned} \tag{3.5}$$

This shows

$$F_{\circledast}(T) = f_*(T)$$

where f_* is defined as the dual of the pullback on forms in 1.4, ([4]).

Now let

$$f: \mathcal{X} \to \mathcal{Y} \tag{3.6}$$

be a C^∞ submission. Hence there is a fibre integral f_* on C^∞ forms and the pullback on the currents f^*. Let

$$F \subset \mathcal{X} \times \mathcal{Y}$$

be its graph. Let ϕ be a test form on $\mathcal{X}$.

$$\begin{aligned}
&\int_{F^{\circledast}(T)} \phi = \int_{(P_{\mathcal{X}})_*[F\wedge(\mathcal{X}\times T)]} \phi \\
&= \lim_{\epsilon\to 0}\int_F R_\epsilon^{\mathcal{X}\times\mathcal{Y}}(\mathcal{X}\times T)\wedge (P_{\mathcal{X}})^*(\phi) \\
&\text{(by Proposition 1.5, the projection formula)} \\
&= \lim_{\epsilon\to 0}\int_F (P_{\mathcal{Y}})^*(R_\epsilon^{\mathcal{Y}}(T))\wedge (P_{\mathcal{X}})^*(\phi).
\end{aligned} \tag{3.7}$$

Notice

$$P'_{\mathcal{Y}} : F \quad\rightarrow\quad \mathcal{Y} \tag{3.8}$$

is isomorphic to the submersion f. Then we apply the fibre integral of $P'_{\mathcal{Y}}$ to have

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_F (P_{\mathcal{Y}})^*(R_\epsilon^{\mathcal{Y}}(T))\wedge (P_{\mathcal{X}})^*(\phi) \\
&= \lim_{\epsilon\to 0}\int_{\mathcal{Y}} R_\epsilon^{\mathcal{Y}}(T)\wedge f_*(\phi) \\
&= \int_T f_*(\phi).
\end{aligned} \tag{3.9}$$

Thus

$$F^{\circledast}(T) = f^*(T). \tag{3.10}$$

We complete the proof. □

Proposition 3.4. *Let $\mathcal{X},\mathcal{Y}$ be two compact manifolds.*
Let

$$F \in C(\mathcal{X}\times\mathcal{Y}) \tag{3.11}$$

be a homogeneous closed, Lebesgue current.

(a) Let T be a Lebesgue current of $\mathcal{X}$ or $\mathcal{Y}$. Then supp($F_{\circledast}(T)$) is contained in the set

$$P_{\mathcal{Y}}\bigg(supp(F)\cap\big(supp(T)\times\mathcal{Y}\big)\bigg);$$

supp($F^{\circledast}(T)$) is contained in the set

$$P_{\mathcal{X}}\bigg(supp(F)\cap\big(\mathcal{X}\times supp(T)\big)\bigg).$$

(b) If T_1, T_2 are Lebesgue and closed (resp. homologous to zero) in $\mathcal{X}$ and $\mathcal{Y}$ respectively, then $F_{\circledast}(T_1), F_{\circledast}(T_2)$ are also closed (resp. homologous to zero).

Proof. (a) Let S be a Lebesgue current on $\mathcal{X} \times \mathcal{Y}$. Let $\mathbf{a} \notin P_{\mathcal{Y}}(supp(S))$. Then there is a neighborhood $B_{\mathbf{a}} \subset \mathcal{Y}$ of $\mathbf{a}$, such that

$$(\mathcal{X} \times B_{\mathbf{a}}) \cap supp(S) = \varnothing.$$

Then for any $\phi \in \mathscr{D}(\mathcal{Y})$ supported in $B_{\mathbf{a}}$,

$$\int_F (P_{\mathcal{Y}})^*(\phi) = 0. \tag{3.12}$$

Then

$$\mathbf{a} \notin supp((P_{\mathcal{Y}})_*(S)).$$

So

$$supp((P_{\mathcal{Y}})_*(S)) \subset P_{\mathcal{Y}}(supp(S)). \tag{3.13}$$

Similarly

$$supp((P_{\mathcal{X}})_*(S)) \subset P_{\mathcal{X}}(supp(S)). \tag{3.14}$$

Now we apply the S to the intersection of currents. Applying part (1), property 1.3,

$$\begin{array}{c} supp((P_{\mathcal{Y}})_*[F \wedge (T \times \mathcal{Y})]) \\ \cap \\ P_{\mathcal{Y}}\Big(supp(F) \cap (supp(T) \times \mathcal{Y})\Big) \end{array} \tag{3.15}$$

The proof of

$$supp(F^{\circledast}(T)) \subset P_{\mathcal{X}}\Big(supp(F) \cap (\mathcal{X} \times supp(T))\Big). \tag{3.16}$$

is similar.

(b) By Property 1.3, the currents

$$[F \wedge (T_1 \times \mathcal{Y})],\ [F \wedge (\mathcal{X} \times T_2)]$$

are closed. Therefore $F^{\circledast}T_2, F_{\circledast}T_1$ are closed. If they are homologous to zero, then by Property 1.3,

$$[F \wedge (T_1 \times \mathcal{Y})],\ [F \wedge (\mathcal{X} \times T_2)]$$

are homologous to zero in $\mathcal{X}, \mathcal{Y}$. Thus $F^{\circledast}T_2, F_{\circledast}T_1$ are homologous to zero.

We complete the proof

$\square$

Example 3.5. *Let X, Y be two smooth projective varieties over $\mathbb{C}$,*

$$f : X \dashrightarrow Y$$

be a rational map. Then there is graph

$$F \subset X \times Y. \tag{3.17}$$

Once $X \times Y$ is equipped with de Rham data (which does not have any requirements for X, Y), there are homomorphisms $F_{\circledast}, F^{\circledast}$

$$\begin{array}{rcl} F_{\circledast} : \mathscr{L}_C(X) & \to & \mathscr{L}_C(Y) \\ F^{\circledast} : \mathscr{L}_C(Y) & \to & \mathscr{L}_C(X). \end{array} \tag{3.18}$$

We denote $F_{\circledast}, F^{\circledast}$ by the more direct notations

$$f_{\circledast}, f^{\circledast}$$

respectively. When $\mathscr{L}_C(X), \mathscr{L}_C(Y)$ are reduced to cohomology, $f_{\circledast}, f^{\circledast}$ are reduced to the usual cohomological correspondences f_*, f^*.

3.2 Functoriality

Due to the dependence of de Rham data, the current's intersection does not rise to the category. However the functoriality not only plays an important role in technique, but also indispensable in idea. So we introduce the general categorical environment where the real intersection theory should fit.

Definition 3.6. *Let k be a whole number. Let X be a smooth projective variety over $\mathbb{C}$. Define $\mathcal{N}_k\mathscr{L}(X)$ to be the linear span of Lebesgue currents*

$$T \in \mathscr{L}(X)$$

satisfying $supp(T)$ lies in an algebraic set A of codimension

$$\geq \frac{codim(T) - k}{2}.$$

(1) A current in $\mathcal{N}_k\mathscr{L}(X)$ will be called $\mathcal{N}_k$ leveled, and k is called current-level. So $\mathcal{N}_k\mathscr{L}$ is a category whose objects are groups $\mathcal{N}_k\mathscr{L}(X)$ with some X, whose morphisms are group homomorphisms.

(2)

$$\mathcal{N}_k\mathscr{L}_C(X) = \mathscr{L}_C(X) \cap \mathcal{N}_k\mathscr{L}(X)$$

also form a filtration.

(3) Let $\mathcal{B}(X)$ be the subgroup of C^∞ singular cycles. Then

$$\mathcal{N}_k\mathcal{B}(X) = \mathcal{B}(X) \cap \mathcal{N}_k\mathscr{L}(X)$$

the subgroups of singular cycles form a filtration.

(4) Let $\mathcal{E}(X) \subset L(X)$ be the subgroup of exact chains. Then

$$\frac{\mathcal{N}_k\mathcal{B}(X) + \mathcal{E}(X)}{\mathcal{E}(X)}$$

is the quotient group whose tensor product with $\mathbb{Q}$ is denoted by

$$\mathcal{N}_k H(X; \mathbb{Q})$$

where k is called class-level.

Definition 3.7. *In the definition of a category for the category theory, we retain all items, but remove the axioms of the associativity and the identity. The remaining collection is called a precategory.*

Example 3.8. *Let $\mathcal{C}o$ be the collection of a C^∞ compact manifold endowed with de Rham data, and a Lebesgue current on a Cartesian product of manifolds. Then the pair of a manifold and a de Rham data is called an object. A Lebesgue current is called a morphism. As usual, $\mathbf{ob}(\mathcal{C}o)$ denotes the collection of objects, $\mathbf{hom}(\mathcal{C}o)$ the collection of morphisms. Let $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$ be three objects in $\mathbf{ob}(\mathcal{C}o)$. Let f_1, f_2 be morphisms in $\mathbf{hom}(\mathcal{X}, \mathcal{Y})$ and $\mathbf{hom}(\mathcal{Y}, \mathcal{Z})$ respectively. Then we define $f_2 \circ f_1 \in \mathbf{hom}(\mathcal{X}, \mathcal{Z})$ to be the current*

$$(P_{\mathcal{X}\mathcal{Z}})_*[(\mathcal{X} \times f_2) \wedge (f_1 \times \mathcal{Z})] \tag{3.19}$$

where $P_{\mathcal{X}\mathcal{Z}} : \mathcal{X} \times \mathcal{Y} \times \mathcal{Z} \to \mathcal{X} \times \mathcal{Z}$ is the projection. This defines the precategory $\mathcal{C}o$.

Example 3.9. *For a precategory, the objects are smooth projective varieties over $\mathbb{C}$ endowed with de Rham data. Morphisms are finite correspondences ([5]). Let the composition be the composition of finite correspondences. We denote this precategory by Cor. But furthermore it is also a category in the usual sense. We should note Cor is not the category of finite correspondences which is originally defined by Voevodsky, and denoted by $Cor_{\mathbb{C}}$. But they are closely related as that Cor is the extension of $Cor_{\mathbb{C}}$ to the C^∞ environment.*

Example 3.10. *For the precategory category, the objects are smooth projective varieties over $\mathbb{C}$ endowed with de Rham data. Morphisms are algebraic cycles on the Cartesian product. Let X, Y, Z three objects, f_1, f_2 are algebraic cycles on $X \times Y$ and $Y \times Z$ respectively. Recall that between two algebraic cycles α, β in an ambient smooth projective variety, the cycle-intersection $\alpha \bullet \beta$ is defined as the sum*

$$\sum_j m_j W_j.$$

where W_j are all irreducible subvarieties, m_i are intersection multiplicities at W_j if W_j are components with the proper dimension in the intersection scheme $|\alpha| \cap |\beta|$, and zero otherwise. Define $f_2 \circ f_1$ to be the algebraic cycle by the cycle-intersection $\bullet$ as

$$(P_{XZ})_*((X \times f_2) \bullet (f_1 \times Z)) \tag{3.20}$$

where $P_{XZ} : X \times Y \times Z \to X \times Z$ is the projection. We denote this precategory by $CCor$. Furthermore $CCor$ is also a usual category.

Due to the presence of de Rham data, Categories $Cor, CCor$ are not in the environment of algebraic geometry. But the connection to algebro-geometric categories is the source of our application.

Let X be a smooth projective variety over $\mathbb{C}$, $\mathscr{Z}(X)$ be the Abelian group freely generated by subvarieties of X. Then by Theorem 2.5,

$$\mathscr{Z}_{\mathbb{R}}(X) := \mathscr{Z}(X) \otimes \mathbb{R}$$

is a subgroup of $\mathscr{L}_C(X)$, and it is a subcategory of the Abelian category. The cycle-intersection is extended to the real coefficients. Then due to the associativity of the cycle-intersection, there is a variant functor

$$\mathcal{F}_c : Cor_{\mathbb{C}} \to \mathscr{Z}_{\mathbb{R}}.$$

Next we define another variant functor for the category, but through the precategory Cor. For the finite correspondence, there is a currents' correspondence

$$F_{\circledast} : \mathscr{L}_C(X) \to \mathscr{L}_C(Y).$$

By Theorem 2.5, its restriction to $\mathscr{Z}_{\mathbb{R}}(Y)$ is a homomorphism $F_{\circledast}|_{\mathscr{Z}_{\mathbb{R}}(Y)}$ independent choice of the de Rham data. Furthermore it coincides with the proper intersection of algebraic cycles. Hence the restriction defines another variant functor

$$\mathcal{F}_{\circledast} : Cor_{\mathbb{C}} \to \mathscr{Z}_{\mathbb{R}}.$$

Then

Proposition 3.11.

$$\mathcal{F}_{\circledast} = \mathcal{F}_c.$$

Proof. Let X, Y be two smooth projective varieties over $\mathbb{C}$, and F a finite correspondence. Then for any $\sigma \in \mathscr{Z}(X)$, the intersection

$$F \cap (|\sigma| \times Y)$$

is proper. By Theorem 2.5, with any de Rham data on X, Y

$$[F \wedge (\sigma \times Y)] = F \bullet (\sigma \times Y) \tag{3.21}$$

where the left hand side is the currents' intersection for the functor $\mathcal{F}_{\divideontimes}$, and right hand side is the current of cycle-intersection for the functor $\mathcal{F}_c$. Hence

$$\mathcal{F}_{\divideontimes} = \mathcal{F}_c.$$

□

3.3 Family of currents

Definition 3.12.

Let S and $\mathcal{X}$ be manifolds endowed with de Rham data. Let $S \times \mathcal{X}$ be endowed with the product de Rham data. Let $\mathcal{I} \in \mathcal{C}(S \times \mathcal{X})$ be a homogeneous Lebesgue current. Let $P_{\mathcal{X}}$ be the projection

$$S \times \mathcal{X} \to \mathcal{X}.$$

We denote the current-correspondence,

$$\mathcal{I}_{\divideontimes}(\{s\}) \tag{3.22}$$

by $\mathcal{I}_s$. The set $\{\mathcal{I}_s\}_{s \in S}$ will be called a family of of currents parametrized by S, and each member $\mathcal{I}_s$ the fibre of $\mathcal{I}$.

Remark The family $\mathcal{I}_s$ depends on extrinsic de Rham data which is not reflected in the notation.

Proposition 3.13. *(current - homotopy) Let $\mathcal{X}$ be a manifold. Let $I_\epsilon \subset \mathbb{R}$ be diffeomorphic to a finite closed interval of $\mathbb{R}$ with two end points 0 and $\epsilon > 0$. Let $\mathbb{R}$ be equipped with a de Rham data, $\mathbb{R} \times \mathcal{X}$ with the product de Rham data. Let $\mathcal{J}$ be a Lebesgue current on*

$$\mathbb{R} \times \mathcal{X}. \tag{3.23}$$

Assume $d\mathcal{J}$ is also Lebesgue. Then

$$\mathcal{J}_\epsilon - \mathcal{J}_0 = \epsilon d\bigg((P_{\mathcal{X}})_*[\mathcal{J} \wedge (I_1 \times \mathcal{X})]\bigg) - \epsilon (P_{\mathcal{X}})_*[d\mathcal{J} \wedge (I_1 \times \mathcal{X})]. \tag{3.24}$$

where $P_{\mathcal{X}} : \mathbb{R} \times \mathcal{X} \to \mathcal{X}$ is the projection. Furthermore, if $\mathcal{J}$ is closed, $\mathcal{J}_\epsilon, \mathcal{J}_0$ are closed and homotopic.

Proof. Apply the Leibniz rule (1.10) to the current

$$\Big[\mathcal{J} \wedge (I_\epsilon \times \mathcal{X})\Big].$$

Then the proposition follows. □

Corollary 3.14. *Let S and $\mathcal{X}$ be manifolds endowed with de Rham data. Let $S \times \mathcal{X}$ be endowed with the product de Rham data. Let $\mathcal{I} \in \mathscr{L}_C(S \times \mathcal{X})$ be a homogeneous closed Lebesgue current. Then the currents $\mathcal{I}_s$ in the family are closed and they have the same cohomology class.*

Proof. Let s_1, s_2 are two points of S. Let $L \subset S$ be a smooth curve through s_1, s_2. Let L be equipped with a de Rham data, and $L \times \mathcal{L}$ be equipped with the product de Rham data. We consider the containment

$$L \times \mathcal{X} \quad \stackrel{i}{\hookrightarrow} \quad S \times X$$

such that $S \times X$ has the associative de Rham data. Then by Proposition 1.7,

$$i_*[(\{s\} \times \mathcal{X}) \wedge [(L \times \mathcal{X}) \wedge \mathcal{I}]] = [(\{s\} \times \mathcal{X}) \wedge \mathcal{I}] \tag{3.25}$$

Let $\mathcal{I}_L = [(L \times \mathcal{X}) \wedge \mathcal{I}]]$. Then the formula (3.25) implies $(\mathcal{I}_L)_s = \mathcal{I}_s$ for each $s \in S$. We denote $\mathcal{I}_s$ by $\mathcal{I}_s^a$ and $(\mathcal{I}_L)_s^a$ to emphasize they both are dependent of associative de Rham data. So precisely formula (3.25) implies

$$(\mathcal{I}_L)_s^a = (\mathcal{I}_s)^a. \tag{3.26}$$

For the product de Rham data on $S \times \mathcal{X}$, we have the definition $\mathcal{I}_s$ whose expression will be changed to $(\mathcal{I}_s)^p$ to indicate its dependence on product de Rham data. We'll denote the cohomology of a closed current by angle bracket $\langle \cdot \rangle$. Since the kernel of de Rham's regulator is homologous to the diagonal,

$$\langle (\mathcal{I}_s)^a \rangle = \langle (\mathcal{I}_s)^p \rangle. \tag{3.27}$$

Also since current-homotopy, Proposition 3.13,

$$\langle (\mathcal{I}_L)_{s_0}^a \rangle = \langle (\mathcal{I}_L)_{s_1}^a \rangle. \tag{3.28}$$

Combining (3.26)-(3.28), we obtain

$$\begin{aligned} &\langle (\mathcal{I}_{s_1})^p \rangle = \langle (\mathcal{I}_{s_1})^a \rangle \\ &= \langle (\mathcal{I}_L)_{s_1}^a \rangle = \langle (\mathcal{I}_L)_{s_0}^a \rangle \\ &= \langle (\mathcal{I}_{s_0})^p \rangle \end{aligned} \tag{3.29}$$

□

Example 3.15. *In the setting of Definition 3.12, we assume $S, \mathcal{X}$ are smooth projective varieties over $\mathbb{C}$, and $\mathcal{I}$ is a variety such that $\mathcal{I}$ is flat over $S\backslash\{s_0\}$, but not flat over S where s_0 is a point of S. Then in algebraic geometry, we have a family of algebraic cycles in $\mathcal{X}$ parametrized by $S\backslash\{s_0\}$, but over z_0, the cycle does not exist. In real intersection theory, the family of currents $\mathcal{I}_s, s \neq s_0$ exists and is equal to the family of algebraic cycles through the integration. However unlike the case of algebraic geometry, the current $\mathcal{I}_{s_0}$ over s_0 still exists, but the family $\mathcal{I}_s$ may not be continuous at s_0 even in the weak topology.*

Example 3.16. *Let X be a smooth projective variety of dimension n over $\mathbb{C}$. Let T be a closed Lebesgue current representing a non-zero primitive cohomology class in $H^n(X;\mathbb{Q})$. Let*

$$V \subset \mathbf{P}^1 \times X. \tag{3.30}$$

be a Lefschetz pencil in X. Assume $\mathbf{P}^1 \times X$ is equipped with a product de Rham data. Let

$$\mathcal{I} = [(\mathbf{P}^1 \times T) \wedge V]. \tag{3.31}$$

be the intersection current (which is a closed current in $\mathbf{P}^1 \times X$). Then each member in the family $\mathcal{I}_t$ is exact for all $t \in \mathbf{P}^1$. But since the projection of $\mathcal{I}$ is T, $\mathcal{I}$ is not exact.

Example 3.17. *Let $\mathcal{X}$ be a C^∞ manifold and T a non-zero homogeneous Lebesgue current in $\mathcal{X}$. Let $\mathbb{R} \times \mathcal{X}$ be equipped with a product de Rham data. Then $\mathcal{I} = \{0\} \times T$ gives a family of currents by Definition 3.12. Notice $\mathcal{I}_t = 0$ for all t including 0 (by Proposition 3.13), but $\mathcal{I}$ is non-zero.*

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