Leveled Sub-cohomology

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Abstract

We define and study an increasing filtration on the rational cohomology of a smooth projective variety. Through that we'll define the level structure on the cohomology, where the "level" is referred to the level in sub-Hodge structures or coniveau filtrations.

Contents

1 Introduction

Let X be a smooth projective variety over the complex numbers. The total Betti cohomology group

$$H(X;\mathbb{Q})=\sum_i H^i(X;\mathbb{Q})$$

over $\mathbb{Q}$ is a $\mathbb{Q}$ linear space. There are many well-known subgroups such as the convineau filtration ([5])

$$N^pH^q(X)\subset H^q(X;\mathbb{Q}),$$

sub-Hodge structures ([2])

$$L^pH^q(X)\subset H^q(X;\mathbb{Q}),$$

and Hodge filtrations

$$F^pH^q(X)\subset H^q(X;\mathbb{C})$$

over $\mathbb{C}$, etc. They are all functorial on the category $SmProj/\mathbb{C}$ of smooth projective varieties over $\mathbb{C}$. In this paper we are going to re-group them, so that a symmetry induced by the Poincaré duality will emerge. At the meantime they become functorial not only on the category $Corr^0(\mathbb{C})$ which includes $SmProj/\mathbb{C}$, but also on a further category.

We are going to axiomatize a sub-cohomology

$$\mathcal{H}_k(X)$$

of total Betti cohomology $H(X;\mathbb{Q})$, indexed by each whole number k. They will be called leveled sub-cohomology at level k. Then to a pair of leveled sub-cohomologies $\mathcal{H}_k(X),\mathcal{J}_{k'}(X)$, we give a sufficient condition for the intersection number pairing between them to be non-degenerate. The non-degeneracy gives the symmetry mentioned above. We call the non-degeneracy the algebraic Poincaré duality, abbreviated as APD. The primary targets are two non-trivial examples. They are

(1) algebraically leveled filtration $\mathcal{N}_k(X)$ of total cohomology $H(X;\mathbb{Q})$ at level k,

(2) Hodge leveled filtration $\mathcal{M}_k(X)$ of total cohomology $H(X;\mathbb{Q})$ at level k.

They are filtrations of $\mathbb{Q}$-subspaces of the total cohomology $H(X;\mathbb{Q})$. Briefly $\mathcal{N}_k(X)$ is defined to be the linear span of all cohomology classes $\alpha \in H(X;\mathbb{Q})$ supported on an algebraic set of dimension $\frac{k+dim_{\mathbb{R}}(\alpha)}{2}$, and $\mathcal{M}_k(X)$ is defined to be the linear span of $\mathbb{Q}$-subspaces of all sub-Hodge structures of level k. It is known that they form two ascending filtrations on $H(X;\mathbb{Q})$

$$\mathcal{N}_0(X) \subset \mathcal{N}_1(X) \subset \cdots \subset H(X;\mathbb{Q}). \tag{1.1}$$

$$\mathcal{M}_0(X) \subset \mathcal{M}_1(X) \subset \cdots \subset H(X;\mathbb{Q}) \tag{1.2}$$

and

$$\mathcal{N}_k(X) \subset \mathcal{M}_k(X).$$

In this paper we initiate a study of a duality among the leveled sub-cohomology at each level k, which include

(a) APD1, a self duality within $\mathcal{N}_k$,

(b) APD2, a duality between $\mathcal{N}_k$ and $\mathcal{M}_k$,

(c) APD3, a self duality within $\mathcal{M}_k$.

2 Functor of leveled sub-cohomology

Definition 2.1. *(Double functor) Let* **A** *be a category and* **B** *be another category. Let*

$$\eta : \mathbf{A} \rightarrow \mathbf{B} \tag{2.1}$$

be a map equipped with two functors, covariant η_1 and contravariant η_2. We call η a double functor.

For the convenience, without a further explanation, we use X to denote a smooth projective variety of dimension n over $\mathbb{C}$.

Definition 2.2. *Let $Corr(\mathbb{C})$ be the category,*

(a) whose objects are smooth projective varieties over $\mathbb{C}$,

(b) whose morphisms from $X \to Y$ are rational correspondences

$$\langle Z\rangle \in CH(X\times Y;\mathbb{Q})$$

(c) whose compositions are the compositions of correspondences.

It is easy to check the graph of identity map is the identity of the category and the associativity of correspondence is the associativity of the morphism. This should not be compared with $Cor(\mathbb{C})$ of finite correspondences ([6]).

Definition 2.3. *Let $Corr(\mathbb{C}, P)$ be the category, whose objects are pair of $X \in Corr(\mathbb{C})$, a polarization $u \in H^2(X;\mathbb{Q})$, and*

$$Hom_{Corr(\mathbb{C},P)} = Hom_{Corr(\mathbb{C})}.$$

Definition 2.4. *Let $H(\cdot;\mathbb{Q})$ the Betti cohomology of a smooth variety over $\mathbb{C}$. We define a double functor, also denoted by $H(\cdot;\mathbb{Q})$ on $Corr(\mathbb{C})$ as follows.*

(a)

$$\begin{array}{ccc} Corr(\mathbb{C}) & \to & Linear\ spaces/\mathbb{Q}. \\ X & \to & H(X;\mathbb{Q}) \end{array} \tag{2.2}$$

(b) For any morphism $\langle Z\rangle \in CH(X\times Y;\mathbb{Q})$ where Z is an algebraic cycle in $X\times Y$, we let P_X, P_Y be the projections from $X \times Y$ to X, Y respectively. Then there is a morphism,

$$H(Y;\mathbb{Q}) \quad \to H(X;\mathbb{Q}) \tag{2.3}$$

defined by

$$\langle Z\rangle^*(\alpha) = (P_X)_*((1\otimes\alpha)\cup\langle Z\rangle)$$

where $(P_X)_*$ is the integration along the fibre (because P_X is a flat morphism). Notice $(P_X)_*$ coincides with the Gysin homomorphism $(P_X)_!$ induced by P_X. This is the contravariant functor on $H(X;\mathbb{Q})$. Similarly we define another morphism

$$H(X;\mathbb{Q}) \quad \to H(Y;\mathbb{Q}) \tag{2.4}$$

by

$$\langle Z\rangle_*(\alpha) = (P_Y)_*((\alpha\otimes 1)\cup\langle Z\rangle).$$

This is the covariant functor. Thus the cohomology $H(\cdot;\mathbb{Q})$ is a double functor. These two functors on the cohomology usually are not inverse to each other. They operate on different degrees.

Remark Double functor here is the union of two functors on the same object. The push-forward $\langle Z\rangle_*$ is just the pull-back with the transpose, $(\langle Z\rangle^t)^*$.

Notice the cohomology $H(\cdot;\mathbb{Q})$ is commonly known as a contravariant functor on the different category $SmProj/\mathbb{C}$, the smooth projective varieties over $\mathbb{C}$,

$$\begin{array}{ccc} SmProj/\mathbb{C} & \rightarrow & Linear\ spaces/\mathbb{Q}. \\ X & \rightarrow & H(X;\mathbb{Q}) \end{array} \tag{2.5}$$

If coupled with Gysin homomorphism, it is also a double functor.

In the following we define a sub-functor of the cohomology $H(\cdot;\mathbb{Q})$.

$$Corr(\mathbb{C}) \ \rightarrow \ Linear\ spaces/\mathbb{Q}. \tag{2.6}$$

where $Corr(\mathbb{C})$ is the category of correspondences.

Definition 2.5.
Let k be a whole number, a leveled sub-cohomology at the level k is a double functor

$$\mathcal{H}_k(\cdot) : Corr(\mathbb{C}) \ \rightarrow \ Linear\ spaces/\mathbb{Q} \tag{2.7}$$

satisfying
(1)

$$\mathcal{H}_k(\cdot) \subset H(\cdot;\mathbb{Q}). \tag{2.8}$$

(2) For X with $n < k$ where $n = dim_{\mathbb{C}}(X)$,

$$\mathcal{H}_k(X) = H(X;\mathbb{Q}). \tag{2.9}$$

For X with $n \geq k$,

$$\mathcal{H}_k(X) \subset \sum_{r=0}^{r=n-k} H^{2r+k}(X;\mathbb{Q}). \tag{2.10}$$

(3) For each X,

$$\mathcal{H}_k(X)\cap \sum_{r\in[0,k]\cup[2n-k,2n]} H^r(X;\mathbb{Q}) = \sum_{r\in[0,k]\cup[2n-k,2n]} H^r(X;\mathbb{Q}). \tag{2.11}$$

(4) Künneth decomposition: for X, Y in $Corr(\mathbb{C})$,

$$\mathcal{K} : \mathcal{H}_k(X)\otimes_{\mathbb{Q}} \mathcal{H}_{k'}(Y) \ \ \rightarrow \ \ \mathcal{H}_{k+k'}(X\times Y), \tag{2.12}$$

where $\mathcal{K}$ is the Künneth isomorphism.

A cohomology class in $\mathcal{H}_k(X)$, or its representative will be called an $\mathcal{H}_k$ leveled cycle (or class).

Remark The word "level" is due to the condition (3), which is the key. An equivalent notion is the coniveau. However the coniveau will not reveal the duality, called algebraic Poincaré duality defined below.

A functor $\mathcal{H}_k$ on $Corr(\mathbb{C})$ is extended to $Corr(\mathbb{C}, P)$ by adding the polarization u. In the following we identify $\mathcal{H}_k$ on both categories. Let $(X, u)\in Corr(\mathbb{C}, P)$.

Use u^i for the linear map

$$\begin{array}{ccc} H^\bullet(X;\mathbb{Q}) & \rightarrow & H^{\bullet+2i}(X;\mathbb{Q}) \\ \alpha & \rightarrow & \alpha\cup u^i. \end{array} \tag{2.13}$$

In the context, we use the same notation u^i to denote its restrictions. Use V to denote the generic hyperplane section that represents the class u. However u is not a functor.

Definition 2.6. *Let $\mathcal{H}_k$ be a leveled sub-cohomology.*

For any $X\in Corr(\mathbb{C}, P)$, primitive leveled sub-cohomology is defined to be

$$\mathcal{H}_{k,prim}(X) = \mathcal{H}_k(X)\cap\left(\sum_{p\leq n} H^p_{prim}(X;\mathbb{Q}) + \sum_{p>n} u^{2p-n} H^p_{prim}(X;\mathbb{Q})\right).$$

We'll denote

$$\sum_{p\leq n} H^p_{prim}(X;\mathbb{Q}) + \sum_{p>n} u^{p-n} H^{2n-p}_{prim}(X;\mathbb{Q})$$

by

$$H_{prim}(X).$$

Both $\mathcal{H}_{k,prim}(-), H_{prim}(-)$ are not functors of $Corr(\mathbb{C}, P)$.

Remark Notice cycles in $\mathcal{H}_{k,prim}(X)$ for $p > n$ are not the conventional primitive cycles.

Definition 2.7. *Algebraic Poincaré duality (APD)*

(a) Let $\mathcal{H}_k, \mathcal{J}_k$ be two leveled sub-cohomology functors. For each X, if the intersection pairing on

$$\mathcal{H}_k(X) \times \mathcal{J}_k(X). \tag{2.14}$$

is a perfect pairing. We say the algebraic Poincaré duality, abbreviated as APD, holds on these two leveled sub-cohomology functors. By the Poincaré duality this pairing has to be between

$$(\mathcal{H}_k(X) \cap H^i(X;\mathbb{Q})) \times (\mathcal{J}_k(X) \cap H^{2n-i}(X;\mathbb{Q})). \tag{2.15}$$

(b) If the intersection pairing on

$$\mathcal{H}_{k,prim}(X) \times \mathcal{J}_{k,prim}(X). \tag{2.16}$$

is a perfect pairing, we say the primitive APD on $\mathcal{H}_k, \mathcal{J}_k$ holds.

3 Convineau Filtration

Algebraically leveled filtration is a filtration re-grouped from the coniveau filtration. While we review the well-known definitions below, we'll give another description using currents. Recall that in [4], Grothendieck created a filtration $Filt'^p$, called "Arithmetic filtration, as it embodies deep arithmetic properties of the scheme ". This later was referred to as the coniveau filtration.

$$N^p H^{2p+k}(X) = \mathcal{N}_k(X) \cap H^{2p+k}(X;\mathbb{Q}).$$

It is defined as a linear span of kernels of the linear maps

$$H^{2p+k}(X;\mathbb{Q}) \quad \to \quad H^{2p+k}(X - W;\mathbb{Q}) \tag{3.1}$$

for a subvariety W of codimension at least p. This is the cohomological view. In the same paper, Grothendieck immediately interpreted it as a linear span of images of Gysin homomorphisms

$$H^{dim(W)+2p+k-2n}(\tilde{W};\mathbb{Q}) \quad \to \quad H^{2p+k}(X;\mathbb{Q}) \tag{3.2}$$

for a subvariety W of codimension at least p with a smooth resolution $\tilde{W}$. This is a view of mixed Hodge structures ([1]). We'll use another interpretation of the coniveau filtration. It is through currents, which are known to unite both homology and cohomology. Let $\mathcal{D}'(X)$ be the space of currents over $\mathbb{R}$ on X. Let $C\mathcal{D}'(X)$ be its subset of closed currents and $E\mathcal{D}'(X)$ be its subset of exact currents. Then

$$\frac{C\mathcal{D}'(X)}{E\mathcal{D}'(X)} = \sum_i H^i(X;\mathbb{C}). \tag{3.3}$$

There is a restriction map on currents

$$\mathcal{R} : \mathcal{D}'(X) \quad \to \mathcal{D}'(X - W) \tag{3.4}$$

for a subvariety W.

Using the formulas (3.3) and (3.4), we define

$$\mathcal{D}^p H^{2p+k}(X)$$

to be the linear span of classes in $H^{2p+k}(X;\mathbb{Q})$ such that they lie in

$$\frac{C\mathcal{D}'(X) \cap kernel(\mathcal{R})}{E\mathcal{D}'(X) \cap kernel(\mathcal{R})}. \tag{3.5}$$

for some W of codimension at least p.

We have the following description of the coniveau filtration.

Proposition 3.1. *Let X be a smooth projective variety over $\mathbb{C}$. Then*

$$\mathcal{D}^p H^{2p+k}(X) = N^p H^{2p+k}(X). \tag{3.6}$$

It says that the cohomology class α lies in

$$N^p H^{2p+k}(X) \tag{3.7}$$

if and only if it is represented by a current whose support is contained in an algebraic set of codimension at least p.

Proof. By the definition

$$\mathcal{D}^p H^{2p+k}(X) \subset N^p H^{2p+k}(X). \tag{3.8}$$

Let's see the converse.

If $\alpha \in N^p H^{2p+k}(X)$, by Cor. 8.2.8, [1], α is the Gysin image

$$H^{dim(A)+2p+k-2n}(\tilde{W};\mathbb{Q}) \quad \to \quad H^{2p+k}(X;\mathbb{Q}) \tag{3.9}$$

for some algebraic subvariety W of codimension at least p. By the definition of the Gysin homomorphism there is a singular cycle σ in $\tilde{W}$ such that the image of σ under the map

$$\rho : \tilde{W} \quad \to X$$

is Poincaré dual to α. Since the support of the current $\rho_*([\sigma])$ is in W, the cohomology class satisfies

$$\rho_*(\langle\sigma\rangle) \in kernel(\mathcal{R}).$$

Thus the current

$$\rho_*([\sigma])$$

is reduced to an element of

$$\mathcal{D}^p H^{2p+k}(X).$$

This completes the proof.

□

Verification of axioms for leveled sub-cohomology will be completed in section 5.

4 Maximal sub-Hodge structure

Definition 4.1. *Let $\Lambda \subset H^{2p+k}(X;\mathbb{Q})$ be a sub-group. If*

$$\Lambda_{\mathbb{C}} = \Lambda^{p,p+k} \oplus \Lambda^{p+1,p+k-1} \oplus \cdots \oplus \Lambda^{p+k,p}$$

where $\Lambda^{i,j}$ are subspaces of $H^{i,j}(X;\mathbb{C})$. Then $\Lambda_{\mathbb{Q}}$ is said to be a sub-Hodge structure of the Hodge structure on $H^{2p+k}(X;\mathbb{Q})$. Let

$$M^p H^{2p+k}(X)$$

be the linear span of subspaces $\Lambda_{\mathbb{Q}}$ for all sub-Hodge structures

$$\Lambda_{\mathbb{Q}} \subset H^{2p+k}(X;\mathbb{Q}).$$

The index p is called the coniveau, and k is called the level.

Above corollary of Deligne shows

$$N^p H^{2p+k}(X) \subset M^p H^{2p+k}(X).$$

Proposition 4.2. *Let X, Y be two smooth projective varieties over $\mathbb{C}$. Let Z be an algebraic cycle in $X \times Y$ of a pure dimension, and $\langle Z \rangle \in CH(X \times Y)$ be it class in the Chow group. Then $\langle Z \rangle_*$ and $\langle Z \rangle^*$ on the cohomology will preserve the level.*

Proof. The pull-back and push-forward operation on cohomology induced from the correspondence $\langle Z \rangle$, are morphisms of Hodge structures. As it known that the difference between i, j for any (i, j) type of cohomology class will be preserved under any morphism of Hodge structures, the level k is defined to be the maximal difference of i, j for all classes in the sub-Hodge structures. Thus it must be preserved under Z.

□

Verification of axioms for leveled sub-cohomology will be completed in section 5.

5 Examples of leveled sub-cohomology

First we make a general claim. Let $Z \subset X$ be an embedding of a smooth variety Z into another smooth variety X over $\mathbb{C}$. Let K be a smooth subvariety of X such that K, Z intersect transversally at a smooth subvariety W. Let $\omega_{x \subset y}$ denote the cohomology Poincaré dual to the submanifold x in manifold y. Let $j : Z \hookrightarrow X$ be the inclusion map.

Lemma 5.1. *Then*

$$j^*(\omega_{Z\subset X}) = \omega_{W\subset Z}. \tag{5.1}$$

Proof. Because the intersection $W = K \cap Z$ is transversal. The normal bundles satisfying

$$N_{W/Z} \subset N_{Z/X}. \tag{5.2}$$

Furthermore the following diagram commutes

$$\begin{array}{ccc} N_{W/Z} & \overset{\psi}{\hookrightarrow} & N_{Z/X} \\ \downarrow & & \downarrow \\ W & \hookrightarrow & Z. \end{array} \tag{5.3}$$

Let η_Z, η_W be the Thom classes of bundles $N_{Z/X} \to Z$ and $N_{W/Z} \to W$. Then

$$\psi^*(\eta_Z) = \eta_W. \tag{5.4}$$

Let's embed the formula (5.3) into the tubular neighborhoods of $W \subset Z$ and $Z \subset X$. Then formula (5.4) becomes (5.1). This completes the proof.

□

Proposition 5.2.

Let $X, Y \in Corr(\mathbb{C})$. Let V be a hyperplane of the projective space containing X, and u be its Poincaré dual.

(1) The map $\imath^*$

$$\imath^* : H(X;\mathbb{Q}) \to H(X\cap V;\mathbb{Q})$$

induced by the inclusion

$$\imath : X \cap V \hookrightarrow X$$

satisfies

$$\imath^*(\mathcal{H}_k(X)) \subset \mathcal{H}_k(X \cap V). \tag{5.5}$$

(2)

$$u(X) \cup \mathcal{H}_k(X) \subset \mathcal{H}_k(X). \tag{5.6}$$

(3) Let $Y \overset{i}{\to} X$ be a regular map. Denote the Gysin homormorphism by $i_!$. Then

$$i_!(\mathcal{H}_k(Y)) \subset \mathcal{H}_k(X). \tag{5.7}$$

and

$$i^*(\mathcal{H}_k(Y)) \subset \mathcal{H}_k(X).$$

Proof. It suffices to show all these maps are realized by correspondences.

(1) Let $\Delta_{V,X}$ be the subvariety in $(X \cap V) \times X$,

$$\Delta_{V,X} = \{(x,x) : x \in X \cap V\}.$$

Then we have

$$\begin{array}{ccc} \Delta_{V,X} & \subset & (X \cap V) \times X \\ \downarrow & & \downarrow_j \\ \Delta_X & \subset & X \times X. \end{array} \tag{5.8}$$

Then we use lemma 5.1 to obtain that

$$j^*(\omega_{\Delta_X \subset (X \times X)}) = \omega_{\Delta_{V,X} \subset (X \cap V) \times X}. \tag{5.9}$$

Since

$$\omega_{\Delta_X \subset (X \times X)} = \langle \Delta_X \rangle$$
$$\omega_{\Delta_{V,X} \subset (X \cap V) \times X} = \langle \Delta_{V,X} \rangle.$$

(2) Let Δ_V be the diagonal in $X \times X$,

$$\Delta_V = \{(x,x) : x \in V\}.$$

Then

$$\langle \Delta_V \rangle = \langle \Delta_V \rangle \cup u \cup u^*. \tag{5.10}$$

where u^* is the dual. Then we check

$$\langle \Delta_V \rangle^*(\alpha) = u \cup (\alpha)$$

for any cohomology class $\alpha \in H(X)$.

(3) Let
$$G_i \subset Y \times X$$
be the graph of the map i. The Gysin homomorphism in section 7, is Poincaé daul to the induced map on the singular homology

$$H_p(Y) \stackrel{i_*}{\to} H_p(X). \tag{5.11}$$

The homomorphism i_* can be expressed as the map $i_\#$ on singular chains. Next we use simplicial complexes of each space. Let $\mathcal{S}_Y$ be a triangulation of Y. This naturally induces a triangulation $\mathcal{S}_G$ of G_i. Then $i_\#$ is the

$$(P_Y)_\#((c \times X) \cap \mathcal{S})$$

where c is a cycle in $\mathcal{S}_Y$, and X is a complex containing the images of all $\mathcal{S}_Y$. Then we reduce them to homology to obtain that

$$i_* = (P_Y)_*(\langle c\rangle \cap \langle G_i\rangle). \tag{5.12}$$

Applying the Poincaré duality to (5.12), we complete the proof for the Gysin homomorphism. To see the pullbaclk i^*, let ω_{G_i} be the Poincaré dual of G_i in $Y \times X$. Then the pullback $i^*(\beta)$ of the cohomology $\beta \in H^p(X;\mathbb{Q})$ is the same as
$$(P_Y)_*(\omega_{G_i} \cdot (1 \otimes \beta)). \tag{5.13}$$
(This is an assertion for any differentiable map).

□

Proposition 5.3. *Let $\mathcal{H}_\bullet$ be a leveled sub-cohomology.*

$$\mathcal{H}_k \subset \mathcal{H}_{k+1}. \tag{5.14}$$

Proof. Let $X, Y \in Corr(\mathbb{C})$. By the Künneth decomposition

$$K : \mathcal{H}_k(X) \otimes_{\mathbb{Q}} \mathcal{H}_1(Y) \subset \mathcal{H}_{k+1}(X \times Y). \tag{5.15}$$

Notice by the definition the fundamental class $1_Y \in \mathcal{H}_l(Y)$ for any whole number l. In particular
$$1_Y \in \mathcal{H}_1(Y).$$

Thus by (2.12)

$$K(\mathcal{H}_k(X)\otimes\{1_Y\})\subset\mathcal{H}_{k+1}(X\times Y). \tag{5.16}$$

Let

$$\nu^*: H(X\times Y) \quad\to\quad H(X) \tag{5.17}$$

be the restriction of cohomology to $X\times\{y\}$ where $y\in Y$ is a point. By proposition 5.2, ν^* preserves the level of the cycles. Then we have composition

$$\mathcal{H}_k(X) \quad\to\quad \mathcal{H}_{k+1}(X\times Y) \quad\to\quad \mathcal{H}_{k+1}(X). \tag{5.18}$$

In cohomology, the composition is simply the identity map. The proposition is proved.

□

Remark This shows a leveled sub-cohomology is an ascending filtration on cohomology.

Proposition 5.4. *We make a convention that*

$$N^iH^{2i+k}(X)=\begin{cases} N^iH^{2i+k}(X) & for \quad i\in[0,dim(X)-k] \\ 0 & for \quad 2i+k\notin[0,2dim(X)] \\ H^{2i+k}(X;\mathbb{Q}) & for \quad 2i+k\in[0,k]\cup[2dim(X)-k,2dim(X)]\end{cases} \tag{5.19}$$

The following sum of coniveau filtration

$$\sum_{r=-\infty}^{+\infty}N^rH^{2r+k}. \tag{5.20}$$

gives a rise to a leveled sub-cohomology at the level k. We'll name it as algebraically leveled filtration and denote it by $\mathcal{N}_k$. Notice that

$$\mathcal{N}_k(X)\cap H^{2r+k}(X;\mathbb{C})=N^rH^{2r+k}(X).$$

Furthermore applying Cartesian product, we obtain that $\mathcal{N}_\bullet$ forms a filtration of leveled sub-cohomology.

Proof. Fix a whole number k. We consider the map

$$\begin{array}{ccc} Corr(\mathbb{C}) & \to & Linear\ spaces/\mathbb{C} \\ X & \to & \mathcal{N}_k(X). \end{array} \tag{5.21}$$

The morphisms are the restrictions of the double functors. Next we show it is covariant. Let X, Y, W be three projective varieties over $\mathbb{C}$. Let

$$Z_1 \in CH(X \times Y), Z_2 \in CH(Y \times W).$$

Then it is suffices to show the composition criterion,

$$(Z_2 \circ Z_1)_* = \langle Z_2 \rangle_* \circ \langle Z_1 \rangle_* \tag{5.22}$$

where $Z_2 \circ Z_1$ is the composition of the correspondences.

Let $\alpha \in H(X \times Y \times W)$ be a cohomology having a homogeneous degree. It will be sufficient to show the intersection

$$(Z_2 \circ Z_1)_*(\alpha) = \langle Z_2 \rangle_* \circ \langle Z_1 \rangle_*(\alpha). \tag{5.23}$$

We consider the triple cohomological intersection in the variety

$$X \times Y \times W,$$

$$\beta = \langle Z_1 \otimes Y \rangle \cup \langle X \times Z_2 \rangle \cup (\alpha \otimes \langle \times X \times Y \rangle). \tag{5.24}$$

Next we use two compositions of the same projection P_W^{XYW},

$$P_W^{XW} \circ P_{XW}^{XYW}, P_W^{YW} \circ P_{YW}^{XYW} \tag{5.25}$$

where the superscript indicates the domain of the projection, and the subscript indicates the target of the projection. Then using the projection formula, we obtain the left-hand side of (5.18) is

$$(P_W^{XW} \circ P_{XW}^{XYW})_*(\beta) = (P_W^{XYW})_*(\beta), \tag{5.26}$$

the right-hand side of (5.18) is

$$(P_W^{YW} \circ P_{YW}^{XYW})_*(\beta) = (P_W^{XYW})_*(\beta). \tag{5.27}$$

This proves (5.18). So the functor is covariant. Similarly its transpose is also covariant. We conclude that

$$X \to \mathcal{N}_k(X)$$

is a double functor. Next we show both morphisms preserve the level. Let's first consider the pull-back. Let X, Y be any smooth projective varieties over $\mathbb{C}$. Let Z be an algebraic cycle in $X \times Y$ of complex codimension l., and

$$\langle Z \rangle \in CH(X \times Y)$$

be the class in the Chow group. Let $\alpha \in N^q H^{2q+k}(Y)$. Then by Deligne's corollary, there is a subvariety $A \subset Y$ such that α is the Gysin image of

$$H^{dim(A)+2p+k-2n}(\tilde{A}; \mathbb{Q}) \quad \to \quad H^{2p+k}(Y; \mathbb{Q}) \tag{5.28}$$

where $\tilde{A}$ is the smooth resolution of A. By the definition of the Gysin homomorphism there is a singular cycle σ in $\tilde{A}$ such that the image of σ under the map

$$\rho : \tilde{A} \ \ \to Y$$

is Poincaré dual to α. We may assume the intersection

$$Z \cap (X \times A)$$

is proper. Applying the definition in cohomology,

$$\langle Z \rangle_*(\alpha)$$

is zero outside of

$$P_X(Z \cap (X \times A)).$$

This shows

$$\langle Z \rangle_*(\alpha) \in N^{p'} H^{q'}(X; \mathbb{C}).$$

Next we calculate the level p'.

Let $dim(Z) = l$, $dim(X) = m$. Notice $\langle Z \rangle^*$ sends

$$H^{2q+k}(Y; \mathbb{C}) \quad \to \quad H^{2q+k+2l-2m}(X; \mathbb{C}),$$

Suppose α lies in $N^q H^{2q+k}(Y)$. It lies in an algebraic cycle σ_a of complex dimension at most

$$m - q,$$

Choose a cycle Z' that is rationally equivalent to Z such that the intersection of

$$Z' \cap (X \times \sigma_a)$$

is proper. Then the complex dimension of $Z' \cap (X \times \sigma_a)$ is at most

$$m + n - q - l$$

The complex dimension of the algebraic set

$$P_X\Big(supp(Z') \cap (X \times supp(\sigma_a)\Big)$$

is also at most

$$m + n - q - l.$$

Since $\langle Z \rangle^*(\alpha)$ lies in

$$P_X(supp(Z') \cap (X \times supp(\sigma_a)),$$

$\langle Z \rangle^*$ sends $N^q H^{2q+k}(Y)$ to

$$N^{l+q-m} H^{2q+k+2l-2m}(X).$$

Thus the level is k. Since the other morphism is the transpose of the same correspondence, the proof for the push-forward is identical after the change of the order of X and Y.

Thus $\mathcal{N}_k$ is a double functor. The conditions (1), (2) and (4) are obvious. Since any cycle lies in X,

$$N^0 H^k(X) = H^k(X; \mathbb{C}).$$

Because of the hard Lefschetz theorem (7, §0, [3]), for $k < n$, any k cycle lies in a plane section of codimension k. Hence

$$N^{n-k} H^{2n-k}(X) = H^{2n-k}(X; \mathbb{C}).$$

By the Poincaré duality the condition (3) is proved. So the functor is leveled. By Künneth decomposition,

$$\mathcal{N}_k(X) \otimes \mathcal{N}_{k'}(X) \subset \mathcal{N}_{k+k'}(X \times Y).$$

Thus $\mathcal{N}_\bullet$ is a filtration of leveled sub-cohomology.

This completes the proof. □

Proposition 5.5. *Let's have the same convention that*

$$M^iH^{2i+k}(X) = \begin{cases} M^iH^{2i+k}(X) & \textit{for} \quad i \in [0, dim(X)-k] \\ 0 & \textit{for} \quad 2i+k \notin [0, 2dim(X)] \\ H^{2i+k}(X;\mathbb{Q}) & \textit{for} \quad 2i+k \in [0,k] \cup [2dim(X)-k, 2dim(X)] \end{cases} \tag{5.29}$$

The sum of maximal sub-Hodge structure

$$\sum_{r=-\infty}^{+\infty} M^rH^{2r+k}(X). \tag{5.30}$$

is a leveled sub-cohomology at a fixed level k. We name it as Hodge leveled filtration and denote this functor by $\mathcal{M}_k$. Notice

$$\mathcal{M}_k(X) \cap H^{2r+k}(X;\mathbb{C}) = M^rH^{2r+k}(X).$$

$\mathcal{M}_\bullet$ is a filtration of leveled sub-cohomology.

Proof. Note we defined

$$\mathcal{M}_k(X) = \sum_{r=0}^{n-k} M^rH^{2r+k}(X). \tag{5.31}$$

Since the induced homomorphisms are all morphisms of Hodge structures. it gives a double functor. All conditions (1)-(4) in definition 2.4 follow.

By the tensor product of Hodge structures, $\mathcal{M}_\bullet$ is a filtration of leveled sub-cohomology.

□

Example 5.6.

The cohomology $H(\cdot;\mathbb{C})$ itself is a leveled sub-cohomology at all levels. This is the trivial leveled sub-cohomology, whose APD is the Poincaré duality. Also

$$H(\cdot;\mathbb{C}) \subset H(\cdot;\mathbb{C}) \subset \cdots \subset H(\cdot;\mathbb{C})$$

forms the trivial filtration of leveled sub-cohomology.

Example 5.7.

Let $CH^p_{alg}(X)$ *be the Chow group of algebraic cycles algebraically equivalent to zero. Then there is an Abel-Jacobi map*

$$AJ : CH^p_{alg}(X) \quad \rightarrow \quad J^p(X) \tag{5.32}$$

Let J^p_a *be its image. Since it is a sub-torus, the tangent space* TJ^p_a *is contained in* $H^{p-1,p}(X;\mathbb{C})$*. We let*

$$H^{2p-1}_a(X;\mathbb{C}) = TJ^p_a \oplus \overline{TJ^p_a}. \tag{5.33}$$

It was proved in [8], $H^{2p-1}_a(X;\mathbb{C})$ *has a sub-Hodge structure. So there is a subspace*

$$H^{2p-1}_a(X;\mathbb{Q}) \subset H^{2p-1}(X;\mathbb{Q})$$

such that

$$H^{2p-1}_a(X;\mathbb{C}) \simeq H^{2p-1}_a(X;\mathbb{Q}) \otimes \mathbb{C},$$

and

$$H^{2p-1}_a(X;\mathbb{C})$$

is called algebraic part of cohomology. It is known that

$$H^1(X;\mathbb{Q}) = H^1_a(X;\mathbb{Q}), H^{2n-1}(X;\mathbb{Q}) = H^{2n-1}_a(X;\mathbb{Q}).$$

Therefore according to the definition 2.4, the algebraic part of cohomology

$$\sum_{i=odd} H^i_a(\cdot;\mathbb{Q})$$

defined by Murre is a leveled sub-cohomology at level 1.

Actually Murre went further and showed that

$$\sum_{i=odd} H^i_a(\cdot;\mathbb{Q}) = \mathcal{N}_1. \tag{5.34}$$

Example 5.8.

The image of cycle maps,

$$A(\cdot) = \sum_i A^i(\cdot)$$

is a leveled sub-cohomology $\mathcal{N}_0$ *at level* 0*.*

Example 5.9.

Primitive cohomology H_{prim} is not a leveled sub-cohomology.
Primitive leveled sub-cohomology is a not leveled sub-cohomology.
So they are not functors.

Example 5.10. *Let X be a smooth projective variety defined over $\mathbb{C}$. Let $\tau \in Gal(\mathbb{C}/\mathbb{Q})$. Then there is another smooth projective variety X_τ over $\mathbb{C}$ defined by ideal $\tau(I(X))$ where $I(X)$ is the ideal defining X. See [9] for detailed discussion.*

Through the algebraic de Rham cohomology, we obtain the isomorphism

$$\tau : H^\bullet(X;\mathbb{C}) \quad \rightarrow \quad H^\bullet(X_\tau;\mathbb{C}), \tag{5.35}$$

where the τ is induced from the isomorphism of the algebraic de Rham cohomology. So τ is an isomorphism of $\mathbb{C}$ linear spaces, but it is not an isomorphism of the $\mathbb{Q}$ linear spaces.

We call the subgroup

$$A^p_\tau H^{2p+k}(X) \subset H^{2p+k}(X;\mathbb{Q}) \tag{5.36}$$

the relative leveled sub-cohomology at level k, if $A^p_\tau H^{2p+k}(X)$ is the maximal sub-space of $H^{2p+k}(X;\mathbb{Q})$ such that

$$\tau\bigg(A^p_\tau H^{2p+k}(X)\bigg) \subset M^p H^{2p+k}(X) \otimes \mathbb{C}. \tag{5.37}$$

We defined the absolute leveled sub-cohomology $A^p H^{2p+k}(X)$ to be the intersection

$$\cap_{\tau\in Gal(\mathbb{C}/\mathbb{Q})} A^p_\tau H^{2p+k}(X).$$

Let Y be another smooth projective variety over $\mathbb{C}$. Let Z be an algebraic correspondence between X and Y. Then Z_τ will defined an algebraic correspondence between X_τ, Y_τ. Then we have two commutative diagrams

$$\begin{array}{ccc} H(X;\mathbb{C}) & \rightarrow & H(X_\tau;\mathbb{C}) \\ \downarrow_{\langle Z\rangle_*} & & \downarrow_{\langle Z_\tau\rangle_*} \\ H(Y;\mathbb{C}) & \rightarrow & H(Y_\tau;\mathbb{C}) \end{array} \tag{5.38}$$

$$\begin{array}{ccc} H(X;\mathbb{C}) & \rightarrow & H(X_\tau;\mathbb{C}) \\ \uparrow_{\langle Z\rangle^*} & & \uparrow_{\langle Z_\tau\rangle^*} \\ H(Y;\mathbb{C}) & \rightarrow & H(Y_\tau;\mathbb{C}). \end{array} \tag{5.39}$$

These diagram imply that

$$A^pH^{2p+k}(X), A^p_\tau H^{2p+k}(X)$$

both form leveled sub-cohomology. Precisely if we let $\mathcal{A}_{\tau,k}$ be a double functor with

$$\mathcal{A}_{\tau,k}(X) = \sum_{p=-\infty}^{+\infty} A^p_\tau H^{2p+k}(X). \tag{5.40}$$

(use a convention as in $\mathcal{N}_k$) and $\mathcal{A}_k$ be a double functor with

$$\mathcal{A}_k(X) = \sum_{p=-\infty}^{+\infty} A^p H^{2p+k}(X). \tag{5.41}$$

Then they both are k leveled sub-cohomology.

Since τ preserves the Hodge filtration,

$$\mathcal{N}_k \subset \mathcal{A}_k \subset \mathcal{M}_k. \tag{5.42}$$

But for arbitrary $\tau \in Gal(\mathbb{C}/\mathbb{Q})$, there is only

$$\mathcal{N}_k \subset \mathcal{A}_{\tau,k}. \tag{5.43}$$

Notice that

$$\mathcal{N}_0 \subset \sum_i AH^i \subset \mathcal{A}_0 \subset \sum_i Hdg^i,$$

where AH is the space of Deligne's absolute Hodge cycles.

Remark

In the examples, we have the relations

$$\mathcal{N}_k \subset \mathcal{M}_k \subset H(\cdot;\mathbb{Q})$$

and

$$\mathcal{N}_k \subset \mathcal{A}_k \subset \mathcal{A}_{\tau,k} \subset H(\cdot;\mathbb{Q}).$$

Hodge conjecture leads to a question: is $\mathcal{N}_\bullet$ a non trivial, maximal leveled sub-cohomology?

6 APD on leveled sub-cohomology

We consider the category of polarized varieties, $Corr(\mathbb{C}, P)$.

Theorem 6.1. *If the primitive APD on a pair of leveled sub-cohomology holds, then APD holds on the pair .*

Let's start with an easy lemma which may be well-known (see, for instance, (4.6), [10]).

Lemma 6.2. *Let V and Z be two smooth projective varieties over $\mathbb{C}$. Let*

$$i : Z \quad \to \quad V$$

be the inclusion map. Let $\theta \in H^\bullet(V; \mathbb{Q})$ be cohomology class. $\omega_Z \in H^\bullet(V; \mathbb{Q})$ be the Poincaré dual of Z in V. Then

(a)

$$i_! i^*(\theta) = \theta \cdot \omega_Z. \tag{6.1}$$

(b) for any cohomology $\eta \in H^\bullet(V; \mathbb{Q})$, the intersection numbers satisfy

$$(\eta, \omega_Z, \theta)_V = (i^*(\eta), i^*(\theta))_Z. \tag{6.2}$$

Proof. (a) We use de Rham cohomology. Let's denote the de Rham representatives of θ and ω_Z by the same letter θ and ω_Z. Let ϕ be a closed C^∞ form on V. Then it suffices to show

$$\int_V i_! i^*(\theta) \wedge \phi = \int_V \theta\omega_Z \wedge \phi. \tag{6.3}$$

Since both sides of (6.1) equals to

$$\int_Z \theta \wedge \phi$$

we complete the proof of (a).

(b) As above we use de Rham cohomology. Then according part (a) left hand side of (6.2) is

$$\int_Z \eta \wedge \theta.$$

Using the intersection in de Rham cohomology, right hand side of (6.2) is also

$$\int_Z \eta\wedge\theta.$$

This completes the proof of (b)

□

Definition 6.3. *(Plane-sectional decomposition). Let* $\mathcal{H}_k$ *be a leveled sub-cohomology. Let*

$$\mathcal{H}_k = L_0\oplus L_1\cdots\oplus\cdots \tag{6.4}$$

where

$$L_i$$

is a direct sum complement of $ker(u^i|_{\mathcal{H}_k})$ *in the* $ker(u^{i+1}|_{\mathcal{H}_k})$*,*

$$L_i\oplus ker(u^i|_{\mathcal{H}_k}) = ker(u^{i+1}|_{\mathcal{H}_k})$$

where the decomposition is not unique. So for each $X\in Corr(\mathbb{C}, P)$*,* $\mathcal{H}_k(X)$ *will be decomposed into finitely many* $L_i(X)$*.*

Remark The decomposition is not unique.

Proof. of theorem 6.1: Let p, k be two fixed whole numbers. Let $\mathcal{H}_k, \mathcal{J}_k$ be two leveled sub-cohomologies.

We'll use the notations

$$\begin{array}{c}\mathcal{H}_k^i = \mathcal{H}_k\cap H^i(\cdot;\mathbb{Q})\\ \mathcal{J}_k^i = \mathcal{J}_k\cap H^i(\cdot;\mathbb{Q}).\end{array}.$$

Then we apply induction on the dimension of X. When $dim(X)$ is the smallest for p, k, which is $\frac{p+k}{2}$. Then both

$$\mathcal{H}_k^p, \mathcal{J}_k^{2n-p}$$

are back to the usual cohomology $H^\bullet(X;\mathbb{Q})$. By the rational Poincaré duality, the APD holds. Next we assume APD holds for $dim_{\mathbb{C}}(X) < n-1$. Consider the X with $dim_{\mathbb{C}}(X) = n$. It suffices to prove that

$$\mathcal{H}_k^p(X) \stackrel{\mathcal{P}}{\to} (\mathcal{J}_k^{2n-p}(X))^\vee \tag{6.5}$$

is surjective, where $\mathcal{P}$ is the map induced from the intersection form.

Next we consider two cases:

(1) $p > n$. Let's recall our goal: for any given $\alpha \in H^p(X;\mathbb{Q})$, we need to find $\mathcal{H}_k$ leveled α_a such that

$$(\alpha_a, \omega)_X = (\alpha, \omega)_X. \tag{6.6}$$

for all $\omega \in \mathcal{J}_k^{2n-p}(X)$. In this statement we regard the intersection pairing $(\alpha, \bullet)_X$ as an element in $(\mathcal{J}_k^{2n-p}(X))^\vee$.

By the hard Lefschetz theorem, the class

$$\alpha = u \cup \beta. \tag{6.7}$$

where $\alpha \in H^p(X;\mathbb{Q}), \beta \in H^{p-2}(X;\mathbb{Q})$. Then applying lemma 6.2, we have the triple intersection number

$$(\beta, u, \omega)_X = (\beta_Y, \omega_Y)_Y,$$

where Y is a smooth hyperplane section of X and $(\bullet)_Y$ is the restriction of the cohomology to Y. By the induction, since ω_Y is $\mathcal{J}_k$-leveled, there is an $\mathcal{H}_k$ leveled cycle α_Y, such that

$$(\beta_Y, \omega_Y)_Y = (\alpha_Y, \omega_Y)_Y. \tag{6.8}$$

Let $i_!$ be the Gysin homomorphism from

$$H^\bullet(Y;\mathbb{Q}) \to H^{\bullet+2}(X;\mathbb{Q})$$

which maps cycles leveled at k to cycles at the same level. Then applying lemma 6.2 again, we obtain

$$\Big(i_!(\alpha_Y), \omega\Big)_X = (\alpha_Y, \omega_Y)_Y = (\alpha, \omega)_X. \tag{6.9}$$

Thus $i_!(\alpha_Y)$ is the $\mathcal{H}_k$ leveled cycle we are looking for.

(2) $p \leq n$. We need to first decompose $(\mathcal{J}_k^{2n-p}(X))^\vee$. This is originated from the decomposition in definition 6.3, the plane-sectional decomposition,

$$\mathcal{J}_k^{2n-p}(X) = L_0(X) \oplus L_1(X) \cdots \oplus L_{[\frac{p-1}{2}]}(X). \tag{6.10}$$

By the topological Poincaré duality we always have the surjective map

$$\begin{array}{rcl} \mathcal{P} : H^p(X;\mathbb{Q}) & \to & (L_i(X))^\vee \\ \alpha & \to & \alpha \cap (\bullet) \end{array} \tag{6.11}$$

for each $0 \leq i \leq [\frac{p-1}{2}]$.

Due to the definition of $L_i(X)$, the map

$$L_i(X) \stackrel{u^i}{\to} H^{2n-p+2i}(X;\mathbb{Q}). \tag{6.12}$$

is injective. Therefore the dual map which is still denoted by u^i,

$$H^{p-2i}(X;\mathbb{Q}) \stackrel{u^i}{\to} (L_i(X))^\vee \tag{6.13}$$

is surjective.

Hence

$$\oplus_{i=1}^{[\frac{p-1}{2}]} H^{p-2i}(X;\mathbb{Q}) \stackrel{\sum_i u^i}{\to} \oplus_{i=1}^{[\frac{p-1}{2}]} (L_i(X))^\vee \tag{6.14}$$

is also surjective.

Let Y be a smooth subvariety such that

$$[Y] = V^i \cap X, \tag{6.15}$$

where $1 \leq i < n$. Thus Y is also an irreducible, smooth projective variety.

Then for any $\omega \in L_i(X), i \neq 0$, we consider the triple intersection number

$$(\alpha_i, u^i, \omega)_X \tag{6.16}$$

Using lemma 6.2, we obtain that

$$(\alpha_i, u^i, \omega)_X = (\alpha_{i,Y}, \omega_Y)_Y \tag{6.17}$$

where $(\cdot)_Y$ is the restriction of the cohomology to its submanifold Y. Notice ω_Y is the pull-back of ω which must be $\mathcal{J}_k$-leveled and Y has dimension lower than n. By the induction, we obtain a $\mathcal{H}_k$-leveled cycle $\alpha^a_{i,Y}$ in Y such that,

$$(\alpha^a_{i,Y}, \omega_Y)_Y = (\alpha_{i,Y}, \omega_Y)_Y. \tag{6.18}$$

Let $i_!$ be the Gysin homomorphism from

$$H^\bullet(Y;\mathbb{Q}) \to H^{\bullet+2i}(X;\mathbb{Q})$$

which maps cycles leveled at k to cycles leveled at k. Then applying lemma 6.2 again, we obtain

$$\left(i_!(\alpha^a_{i,Y}), \omega\right)_X = (\alpha, u^i, \omega)_X = \psi^i(u^i\omega), \tag{6.19}$$

where $i_!(\alpha^a_{i,Y})$ is $\mathcal{H}_k$-leveled. This show the surjectivity of the map

$$\mathcal{H}^p_k(X) \stackrel{\mathcal{P}}{\to} (\oplus_{i\neq 0} L_i(X))^\vee. \tag{6.20}$$

Now we work with $(L_0(X))^\vee$. Let $\omega \in L_0(X)$ be the testing cycle. As before we consider $\alpha_0 \in H^p(X;\mathbb{Q})$ that represents an element in $(L_0(X))^\vee$.

For any such α_0, there is the Lefschetz decomposition

$$\alpha_0 = \alpha^0_0 + \sum_{l\geq 1} u^l \alpha^l_0. \tag{6.21}$$

Using the same inductive argument above, we obtain a $\mathcal{H}_k$ leveled cycle $\alpha_0(1)$ such that

$$(\alpha_0(1), \omega)_X = (\sum_{l\geq 1}(u^l\alpha_l), \omega)_X, \tag{6.22}$$

for any $\omega \in L_0(X)$. By Lefschetz decomposition

$$\omega = u^{n-p}\omega_p + u^{n-p+1}\omega_{p-2} + \cdots + u^{n-p+[\frac{p}{2}]}\omega_{p-2[\frac{p}{2}]}. \tag{6.23}$$

where $\omega_j \in H^j_{prim}(X;\mathbb{Q})$. Notice $\omega u^i = 0$ for all $1 \leq i \leq [\frac{p-1}{2}]$. Hence all primitive cycles

$$\omega_{p-2} = \omega_{p-3} = \cdots = \omega_{p-2[\frac{p}{2}]} = 0.$$

Therefore

$$\omega = u^{n-p}\omega^p \tag{6.24}$$

where ω^p is primitive. By the assumption of primitive APD (notice $\omega = u^{n-p}\omega^p$ is $\mathcal{J}_k$ leveled), there is a primitive, $\mathcal{H}_k$ leveled cycle $\alpha_0(2)$ such that

$$(\alpha_0(2), \omega)_X = (\alpha^0_0, \omega)_X. \tag{6.25}$$

Thus

$$(\alpha_0(1) + \alpha_0(2), \omega)_X = (\alpha_0, \omega)_X. \tag{6.26}$$

Now we combine all components in the decomposition

$$(\mathcal{J}_k^{2n-p}(X))^\vee = (L_0(X))^\vee \oplus \cdots \oplus (L_{[\frac{p-1}{2}]}(X))^\vee. \tag{6.27}$$

For any element $\psi \in (\mathcal{J}_k^{2n-p}(X))^\vee$, it is decomposed as

$$\sum_{i=0}^{[\frac{p-1}{2}]} \psi^i \tag{6.28}$$

where $\psi^i \in (L_i(X))^\vee$ and ψ^i can be represented through intersection form by the cycles α_i. Then we can find the $\mathcal{H}_k$ leveled cycle

$$\alpha_0(1) + \alpha_0(2) + \sum_{i\neq 0} i_!(\alpha^a_{i,Y}) \tag{6.29}$$

such that its Poincaré dual is ψ. We complete the proof.

□

7 Glossary

(1) If $X \xrightarrow{i} Y$ is a continuous map between two real compact manifolds, then the induced homomorphism $i_!$ in the graph,

$$\begin{array}{ccc} H_p(X;\mathbb{Q}) & \xrightarrow{i_*} & H_p(Y;\mathbb{Q}) \\ {\scriptstyle Poincaré}\Big\Updownarrow{\scriptstyle duality} & & {\scriptstyle Poincaré}\Big\Updownarrow{\scriptstyle duality} \\ H^{dim(X)-p}(X;\mathbb{Q}) & \xrightarrow{i_!} & H^{dim(Y)-p}(Y;\mathbb{Q}) \end{array} \tag{7.1}$$

will be called Gysin homomorphism.

(2) $\mathbb{M}^{p,2p+k}(X)$ is the maximal sub-Hodge structures of coniveau p at level k.

(3) $N^pH^{2p+k}(X)$ is the coniveau filtration of coniveau p at level k.

(4) $Hdg^\bullet(X)$ is the subspace spanned by Hodge classes.

(5) $A^\bullet(X)$ is the subspace of the rational cohomology, spanned by

algebraic cycles.
(6) $a^\vee$ denotes the dual of a vector space if a is a vector space or a vector.
(7) a^* denotes a pullback in various situation depending on the context.
(8) a_* denotes a pushforwad in various situation depending on the context.
(9) $\langle a \rangle$ denotes a classes in various groups represented by an object a.
(10) $(\cdot, \cdot)_X$ is the intersection number in X between a pair of the same or/and different types of objects.
(11) $(\cdot, \cdot, \cdot, ...)_X$ denotes the intersection number among multiple objects.
(12) CH denotes the Chow group, CH_{alg} denotes the subgroup of cycles algebraically equivalent to zero.
(13) J denotes the intermediate Jacobians.
(14) We'll drop the name "Betti" on the cohomology. So all cohomology are Betti cohomology.

References

[1] P. Deligne , *Théorie de Hodge: III*, Publ. Math IHES 44 (1974), pp 5-77.

[2] P. Griffiths, *Mumford-Tate groups*, Lecture note on joint work with M. Green and M. Kerr.

[3] P. Griffiths, J. Harris, *Principles of algebraic geometry*, John Wiley & sons, Inc. 1978.

[4] A. Grothendieck, *Hodge's general conjecture is false for trivial reasons*, Topology, Vol 8 (1969), pp 299-303.

[5] J. Lewis, *Three lectures on the Hodge conjectures*,

[6] C. Mazza, V. Voevodsky, C. Weibel, *Lecture Notes on Motivic Cohomology*, Clay Mathematics Monographs (2000).

[7] J. S. Milne, *Motives—Grothendieck's Dream*, http://jmilne.org/math /xnotes/MOT.pdf, April, 2012.

[8] J. Murre, *Abel-Jacobi equivalence versus incidence equivalence for algebraic cycles of codimension two*, Topology **24** (1985), pp 361-367.

[9] C. Voisin, *Lectures on the Hodge and Grothendieck-Hodge conjectures*, Rend.Sem., Univ. Politec. Torino, Vol. 69, 2(2011), pp 149-198.

[10] S. Zucker, *Generalized intermediate Jacobians and the theorem on normal functions*, Inventiones Mathematica, **33**(1976), pp 185-222