Hilbert scheme of rational curves on a generic hypersurface

B. Wang
(汪 镔)

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Abstract

Let X be a generic hypersurface of degree h in projective space $\mathbf{P}^n, n \geq 4$ over the complex numbers. Let d be a fixed natural number. Let $\mathcal{M}_d(X)$ be the open sub-scheme of the Hilbert scheme, parameterizing irreducible rational curves of degree d on X. In this paper, we show that if $\mathcal{M}_d(X) \neq \varnothing$, then it is smooth and of dimension

$$(n+1-h)d + n - 4 \geq 0. \tag{0.1}$$

The result directly confirms two conjectures of Voisin:
1. if X is Calabi-Yau of dimension at least 3 and very general, then the rational curves on X cover a countable union of Zariski closed subsets of codimension ≥ 2;
2. if X is of general type, then the degree of rational curves on it is bounded.

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Address: Mathematics and computer science department,
Rhode Island college, Providence, RI 02908, USA

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1 Introduction

1.1 Statement

We work with the complex numbers, $\mathbb{C}$. For the statements we use Zariski topology.

Theorem 1.1. (Main theorem)

Let $\mathbf{P}^n$ *be the projective space over* $\mathbb{C}$ *of dimension* $n \geq 4$. *Let* $f \in H^0(\mathcal{O}_{\mathbf{P}^n}(h))$ *be generic, and*

$$X = div(f).$$

Let $\mathcal{M}_d(X)$ *be the open sub-scheme of the Hilbert scheme, parameterizing irreducible rational curves of degree* d *on* X.

If $\mathcal{M}_d(X)$ *is non empty, then*

$$(n+1-h)d+n-4 \geq 0$$

and $\mathcal{M}_d(X)$ *is smooth, of dimension*

$$(n+1-h)d+n-4. \tag{1.1}$$

Theorem 1.1 does not give a complete structure of rational curves on generic hypersurfaces, not even the existence. But it has enough information to confirm Voisin's conjectures in [4]. Theorem 1.1 shows that

Corollary 1.2. *If* X *is a generic hypersurface of* $\mathbf{P}^n$ *of general type, i.e.,* $deg(X) > n+1$, *then the degrees of rational curves* $C \subset X$ *have an upper bound*

$$deg(C) \leq \frac{n-4}{deg(X)-n-1}. \tag{1.2}$$

Voisin conjectured that $deg(C)$ is bounded for all n. But the corollary following from Theorem 1.1 is only valid in the case $n \geq 4$. So to complete Corollary 1.2, we deal with the missing case $n \leq 3$ in section 4.

Notice that a Calabi-Yau hypersurface satisfies $n+1-h=0$. So Theorem 1.1 says $dim(\mathcal{M}_d(X)) = n-4$. We obtain the corollary.

Corollary 1.3. *Let* X *be a very general Calabi-Yau hypersurface of* $\mathbf{P}^n, n \geq 4$. *Then the dimension of a parameter space of a family of rational curves of each degree* d *on* X *is* $\leq n-4$ *provided it is non-empty.*

This confirms Voisin's speculation: rational curves on X cover a countable union of Zariski closed subsets of codimension ≥ 2.

1.2 Outline of the proof

The hypersurfaces in a projective space have three types.
1) Fano, $n+1 > h$,
2) Calabi-Yau, $n+1 = h$,
3) of general type, $n+1 < h$.

We prove the results in two cases accordingly: (I) Calabi-Yau and Fano, (II) of general type, where (II) follows from (I) by the standard technique of deformation theory ([3]). But the proof of the case (I), which is the main proof, is somewhat non standard.[1] It sets up the external data for the matrix algebra to compute the Jacobian matrices of the ALTERNATIVE of the Hilbert scheme.

1.2.1 Calabi-Yau and Fano

• *The setting*

In this section we assume $n+1 \leq h$. The idea, which requires $n+1 \leq h$, is to use the matrix algebra to attack the alternative of the Hilbert scheme. It can be described with the chart:

$$Alternative \xrightarrow{\text{Matrix algebra}} Normal\ sheaf \xrightarrow{\text{Deformation theory}} Hilbert\ scheme. \tag{1.3}$$

The alternative is not new. Let's see the definition.

Let

$$S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$$

be the space of degree h hypersurfaces of $\mathbf{P}^n$. Let

$$M = (H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus n+1}, \tag{1.4}$$

be the affine space of all (n+1)-tuples of homogeneous polynomials in two variables of degree d. The open set M_d of M has the projectivization isomorphic to the Hilbert scheme of regular maps

$$\{[c] \in Hom_{bir}(\mathbf{P}^1, \mathbf{P}^n) : deg([c](\mathbf{P}^1)) = d\},$$

[1]It overcame the same difficulty as that in Clemens' conjecture ([1]).

For the simplicity, we still call $c \in M_d$ a rational curve, and denote the rational map $[c] : \mathbf{P}^1 \to \mathbf{P}^n$ by the same letter c. So M is not the Hilbert scheme of rational curves, but we'll use M_d as an alternative to replace the Hilbert scheme $\mathcal{M}_d$. Then the alternative incidence scheme is defined as follows. Let $\mathbb{P} \subset S$ be a generic 2-dimensional plane. We have the alternative triangle to replace the usual triangle of the Hilbert scheme,

$$\begin{array}{ccc} & \Gamma_{\mathbb{P}} & \\ {\scriptstyle P_l} \swarrow & & \searrow {\scriptstyle P_r} \\ \mathbb{P} & & M_d \end{array} \tag{1.5}$$

where $\Gamma_{\mathbb{P}}$ is the non-empty alternative incidence scheme of the containment relation,

$$\{(f, c) \in \mathbb{P} \times M_d : c(\mathbf{P}^1) \subset f\}$$

and P_l, P_r are the projections with the dominant P_l. Our observation is that over the open set $O_{\mathbb{P}} \subset \mathbb{P}$,

$$\Gamma_{\mathbb{P}} \cap (O_{\mathbb{P}} \times M_d)$$

ought to be scheme-theoretically isomorphic to the projection

$$P_r(\Gamma_{\mathbb{P}} \cap (O_{\mathbb{P}} \times M_d)).$$

(which is not so obvious but reasonable). In particular

$$T_{(f_g, c_g)}\Gamma_{\mathbb{P}} \simeq T_{c_g}\big(P_r(\Gamma_{\mathbb{P}})\big) \tag{1.6}$$

for a point $(f_g, c_g) \in \Gamma_{\mathbb{P}}$ with S-generic f_g, where "S-generic" means the genericity in S. This allows us to change the focus to rational curves $P_r(\Gamma_{\mathbb{P}})$ (which has a fundamental importance). But over a projective subvariety $W \subsetneqq \mathbb{P}$ the incidence scheme

$$\Gamma_W = \Gamma \cap (W \times M_d)$$

is known to be reducible, and not all components dominate W. Since we are only interested in irreducible components dominating W, so we'll use I_W to denote an irreducible component of

$$P_r(\Gamma_W)$$

dominating W. If $W_1 \subset W_2 \subset S$, we always take the components with the containment relation

$$I_{W_1} \subset I_{W_2}.$$

In particular $I_{\{f\}}$ for a point $f \in S$ is abbreviated as I_f. Then the observation (1.6) can be formulated as follows.

Proposition 1.4.

(1) There is an isomorphism

$$T_{(f_g,c_g)}\Gamma_{\mathbb{P}} \simeq T_{c_g} I_{\mathbb{P}}, \tag{1.7}$$

where $(f_g, c_g) \in \Gamma_{\mathbb{P}}$ is a point with S-generic $f_g \in \mathbb{P}$. Denote

$$G_{\mathbb{P}} = \{c \in I_{\mathbb{P}} : (f,c) \in \Gamma_{\mathbb{P}} \text{ and } f \in \mathbb{P} \text{ is generic}\}$$

By the dominance of P_l, (1.7) is equivalent to

$$dim\big(T_{c_g}(I_{f_g})\big) + 2 = dim\big(T_{c_g} I_{\mathbb{P}}\big) \tag{1.8}$$

for $c_g \in G_{\mathbb{P}}$.

(2) Furthermore

$$dim\big(T_{c_g}(I_{f_g})\big) = dim H^0(c_g^*(T_{X_g})) + 1,$$

where $X_g = div(f_g)$.

Remark Part (2) serves as a transition from the alternative to the normal sheaf.

After this proposition our focus is shifted to $I_{\mathbb{P}}$. So we can state the key result.

Proposition 1.5. *Assume all notations as above. For $c_g \in G_{\mathbb{P}}$,*

$$dim(T_{c_g} I_{\mathbb{P}}) = (n+1-h)d + n + 2. \tag{1.9}$$

Remark $G_{\mathbb{P}}$ is a constructible set. So $c_g \in G_{\mathbb{P}}$ may not be generic.

The result of Proposition 1.5 for the alternative can be interpreted in the Hilbert scheme through the normal sheaf over $\mathbf{P}^1$ defined as

$$N_{c_g/X_g} := c_g^*(\mathcal{H}om(\mathcal{I}_{c_g(\mathbf{P}^1)}/\mathcal{I}^2_{c_g(\mathbf{P}^1)}, \mathcal{O}_{\mathbf{P}^1})) \tag{1.10}$$

where $\mathcal{I}_{c_g(\mathbf{P}^1)}$ is the ideal sheaf of the scheme $c_g(\mathbf{P}^1)$. The interpretation is the result in Theorem 1.1, so the normal sheaf serves as the bridge connecting the alternative and Hilbert scheme. More intuitively we'll see that Propositions 1.4, 1.5 imply that the generic fibre of P_l is reduced and has dimension $(n+1-h)d+n$. Considering 4 is the dimension of the automorphism group of an irreducible rational curve, we obtain the main theorem.

We would like to point out that all propositions can be reformulated in Hilbert schemes, however the technique of proof can't.

•• *The computation – Differentials of the holomorphic map*

Proposition 1.4 is straightforward. So we focus on Proposition 1.5 which is not straightforward. We'll formulate it as a calculation of the surjectivity of the differentials of holomorphic maps. In representation, the sujectivity is the non-degeneracy of a Jacobian matrix associated to the differential. We'll use two holomorphic maps, one of which is called the direct holomorphic map ν_1, the other is called indirect holomorphic map ν_2. They are all extrinsic with respect to the intrinsic Proposition 1.5. Let's define ν_1. Let $\mathbb{P}$ be any 2-dimensional plane in S spanned by three non zero points f_0, f_1, f_2. Choose generic $hd+1$ points $t_i \in \mathbf{P}^1$ (generic in $Sym^{hd+1}(\mathbf{P}^1)$), and let

$$\mathbf{t} = (t_1, \cdots, t_{hd+1}) \in Sym^{hd+1}(\mathbf{P}^1).$$

We call $\mathbf{t}$ or $t_i, i = 1, \cdots, hd+1$ the designated points. In the rest of the paper, we'll use the following evaluations in affine coordinates:

(a) fix an affine open set $\mathbb{C} \subset \mathbf{P}^1$, and use t_i or t to denote a complex number in $\mathbb{C}$,

(b) fix an affine space $\mathbb{C}^{n+1}$ such that $\mathbf{P}(\mathbb{C}^{n+1}) = \mathbf{P}^n$, and for $c \in M_d$, use $c(t)$ to denote the following image

$$\begin{array}{ccc} \mathbf{P}^1 & \stackrel{c}{\rightarrow} & \mathbf{P}^n \\ \uparrow & & \uparrow \\ \mathbb{C} & \stackrel{c|_{\mathbb{C}}}{\rightarrow} & \mathbb{C}^{n+1}\backslash\{0\} \\ \Cup & & \Cup \\ t & \rightarrow & c(t), \end{array}$$

(c) a hypersurface f is a homogeneous polynomial of degree h in $n+1$ variables, i.e. f is a holomorphic function on $\mathbb{C}^{n+1}$.

Now we we use evaluations in (a), (b), (c) to define polynomials $b_i(c)$ in M as

$$b_i(c) = \begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} \tag{1.11}$$

for $i = 3, \cdots, hd+1$, where $c \in M$ and $|\cdot|$ denotes the determinant.

Using these polynomials, we define a holomorphic map

$$\begin{array}{rcc} \nu_1 : M & \rightarrow & \mathbb{C}^{hd-1} \\ c & \rightarrow & (b_3(c), b_4(c), \cdots, b_{hd+1}(c)). \end{array} \tag{1.12}$$

The representation of the Jacobian matrix of the differential $(\nu_1)_*$ at a point c_g depends on numerous variables, which we call the Jacobian data

Definition 1.6. *(Jacobian data). We defined the Jacobian data to be the collection of following choices. Intrinsic: $\mathbb{P}$ and a point $c_g \in I_{\mathbb{P}}$; Extrinsic: a specific basis $\{f_0, f_1, f_2\}$, the distinct designated points $\mathbf{t}$ of $\mathbf{P}^1$, the order of $\mathbf{t}$, local coordinates of M_d, and affine open sets for evaluations in (a), (b), (c).*

Jacobian data is central to the proof. While the surjectivity ν_1 depends on the intrinsic variables only, but the representation of the Jacobian matrix heavenly depends on extrinsic data. The specialization of Jacobian data leads to a representation of the Jacobian matrix showing that

Proposition 1.7. *If $n+1 \leq h$, then $(\nu_1)_*$ is surjective at $c_g \in G_{\mathbb{P}}$ for generic $\mathbb{P}$.*

Proposition 1.7 is the result of manipulating the Jacobian data. For its application, we should note that Proposition 1.5 follows from Proposition 1.7 because for generic $\mathbb{P}$, the Zariski tangent space $T_{c_g} I_{\mathbb{P}}$ is the kernel of $(\nu_1)_*$. Let's see the reason. The incidence scheme

$$\Gamma_{\mathbb{P}} \subset \mathbb{P} \times M_d \tag{1.13}$$

is defined by $hd+1$ polynomial equations

$$f(c(t_1)) = \cdots = f(c(t_{hd+1})) = 0 \tag{1.14}$$

for the variables $(f, c) \in \mathbb{P} \times M_d$. Then the resultants of the polynomials (in f, c) in (1.14) after eliminating the linear variable f are

$$\begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_j)) & f_1(c(t_j)) & f_0(c(t_j)) \\ f_2(c(t_k)) & f_1(c(t_k)) & f_0(c(t_k)) \end{vmatrix} \tag{1.15}$$

for $1 \leq i, j, k \leq hd+1$, that define the projection $P_r(\Gamma_{\mathbb{P}})$. To calculate the Zariski tangent space of $P_r(\Gamma_{\mathbb{P}})$, in the following we restrict (1.15) to a local analytic neighborhood to remove extraneous equations. Since $\mathbb{P}$ is generic, by Proposition 2.3 which will be proved below, the generic $\mathbb{P}$ satisfies **Pencil condition** (for $\mathbb{P}$): for a generic $f \in \mathbb{P}$, $I_f \cap I_g = \varnothing$ for any other $g \in \mathbb{P}$.

Let us continue with the pencil condition. Let $c_g \in G_{\mathbb{P}}$ ($I_{\mathbb{P}}$ is non-empty) be a rational curve. If the subspace

$$\Lambda_{c_g} = span\left\{ \left(f_2(c_g(t)), f_1(c_g(t)), f_0(c_g(t)) \right) \right\}_{t \in \mathbf{P}^1}$$

in $\mathbb{C}^3$ had dimension one. Then there would be generic vectors (genericity is due to the genericity of the hypersurface containing c_g) $\beta_i, i = 1, 2$ in $\mathbb{C}^3$ such that

$$\beta_i \cdot \Lambda_{c_g} = 0$$

where $\cdot$ is the "dot" product in $\mathbb{C}^3$. Hence there are two generic hypersurfaces in the collection $\mathbb{P}$,

$$\beta_1 \cdot (f_2, f_1, f_0), \qquad and \qquad \beta_2 \cdot (f_2, f_1, f_0)$$

and both contain c_g. Therefore the pencil condition is violated. So $dim(\Lambda_{c_g}) \geq 2$ (actually it can't be 3 because $c_g \in I_{\mathbb{P}}$). Thus we obtain two linearly independent 3-dimensional vectors

$$\begin{pmatrix} f_2(c(t_1)), f_1(c(t_1)), f_0(c(t_1)) \\ f_2(c(t_2)), f_1(c(t_2)), f_0(c(t_2)) \end{pmatrix}$$

for all c in a sufficiently small analytic open set U_{M_d} in M_d, centered around c_g. They span the plane Λ_c (depending on c) in $\mathbb{C}^3$. Then if

$$b_i(c) = 0 \ for \ i = 3, \cdots, hd+1,$$

at some c in the neighborhood, all $hd+1$ vectors

$$\left(f_2(c(t_i)), f_1(c(t_i)), f_0(c(t_i)) \right), \qquad i = 1, \cdots, hd+1$$

must lie in the plane Λ_c. This implies that polynomials of (1.15) vanish at the same c. Thus if we let $U_{\mathbb{P}} = U_{M_d} \cap I_{\mathbb{P}}$ be the restriction of $I_{\mathbb{P}}$ to the analytic neighborhood, then it is only defined by $hd-1$ equations

$$b_i(c) = 0, i = 3, \cdots, hd+1.$$

Thus

$$ker((\nu_1)_*|_{c_g}) = T_{c_g} I_{\mathbb{P}}, \tag{1.16}$$

for $c_g \in G_{\mathbb{P}}$. By the surjectivity of $(\nu_1)_*$ in Proposition 1.7, the kernel of $(\nu_1)_*$ at a point of $G_{\mathbb{P}}$ for a generic $\mathbb{P}$ has dimension $(h+1-n)d+n+2$. We proved Proposition 1.5.

At last we mention the proof of Proposition 1.7, i.e. the surjectivity of $(\nu_1)_*$. The proof is simply a specialization of Jacobian data. But there is one trick before the specialization. We'll add 6 components to extend the direct ν_1 to the indirect holomorphic map $\nu_2 : M \to \mathbb{C}^{hd+5}$, surjectivity of whose differential implies the surjectivity of $(\nu_1)_*$ at the same point. Next we divide the Jacobian matrix of $(\nu_2)_*$ to 4 blocks. For each block, we specialize the corresponding variables in Jacobian data separately to obtain the non-degeneracy of the block. Then we deform variables to a general position to unite the blocks to a non degenerate Jacobian matrix.

1.2.2 General type

A generic hypersurface of general type is a plane section X of a generic Calabi-Yau hypersurface Y. Then a rational curve $C \subset X$ is automatically a rational curve on Y. Applying the projection $T_\bullet Y \to T_\bullet X$, we obtain that $H^1(N_{C/Y}) = 0$ implies $H^1(N_{C/X}) = 0$. Since in the Calabi-Yau case we have $H^1(N_{C/Y}) = 0$, then $H^1(N_{C/X}) = 0$. A general deformation theory says H^0 of the normal sheaf is the tangent space of the Hilbert scheme, and H^1 contains the obstruction space. Hence the deformation of C in X is free of obstruction. This implies the Main theorem in this case.

1.3 Organization

The rest of the paper is organized as follows. In section 2, we prove Proposition 1.4. It gives the principle idea: shift the focus from Hilbert scheme to the alternative for rational curves. In section 3, we use the Jacobian data to study the cases of Calabi-Yau and Fano. In section 4, we use the standard technique in deformation theory to prove the result for hypersurfaces of general type.

2 First order deformation of rational curves

In this section we try to understand the deformation of the pair (f, c). The main purpose is to show how to use the deformation of the pair to change our focus from the pairs to the rational curves only. The results hold for all types of generic hypersurfaces, but we'll only use them for Calabi-Yau and Fano.

2.1 First order deformations of the pair

Lemma 2.1. *Consider the maps in the usual triangle of the Hilbert scheme,*

$$\begin{array}{ccc} & \Gamma & \\ {\scriptstyle P_l'} \swarrow & & \searrow {\scriptstyle P_r'} \\ S & & \mathcal{M}_d(\mathbf{P}^n) \end{array} \tag{2.1}$$

where $\mathcal{M}_d(\mathbf{P}^n)$ is the Hilbert scheme of irreducible rational curves of degree d in $\mathbf{P}^n$, Γ is the non-empty incidence scheme of the containment relation,

$$\{(f, C) \in S \times \mathcal{M}_d(\mathbf{P}^n) : C \subset f\}.$$

Assume the projection P_l' is dominant. Then at a point $(f, C) \in \Gamma$ with S-generic f, the homomorphism

$$(P_l')_* : T_{(f,C)}\Gamma \quad \to \quad T_f S \tag{2.2}$$

is surjective.

Proof. We divide the components of Γ into two types: (I) Γ_1 the collection of components dominating S, (II) Γ_2 the collection of components not dominating S. Then $P_l'(\Gamma_2)$ is a lower dimensional subvariety of S. Let

$$E = \{(f, C) \in \Gamma_1 : (P_l')_*(T_{(f,C)}\Gamma) \neq T_f S\}$$

be the subset which is a subvariety. If $P_l'|_E : E \to S$ dominated S, then there was a smooth, non empty open set U_E of E, to which the restriction of $P_l'|_E$ was smooth. Thus its differential must've been surjective. This would've contradicted the definition of E. Hence $P_l'(E)$ is also a lower dimensional subvariety of S. Then for S-generic $f \in P_l'(\Gamma_1 \backslash E)$,

$$T_{(f,C)}\Gamma \quad \to \quad T_f S \tag{2.3}$$

is surjective. This proves the lemma

□

Lemma 2.1 holds if we change the Hilbert scheme $\mathcal{M}_d$ to M_d.

Definition 2.2. *We apply this lemma to obtain a special notation for the first oder deformation of the rational curves. Let $(f_0, C_0) \in \Gamma$ be a point satisfying Lemma 2.1. Let c_0 be the normalization of C_0. Let $f \in S$ be another hypersurface. We denote a vector parallel to the line through f_0, f by $\overrightarrow{f}$. Then the lemma implies that there is a vector $\langle \overrightarrow{f} \rangle_M$ in $T_{c_0}M$ such that*

$$(\overrightarrow{f}, \langle \overrightarrow{f} \rangle_M)$$

lies in the tangent space of the alternative incidence scheme. We denote the corresponding section in $H^0(c_0^*(T_{\mathbf{P}^n}))$ by

$$\langle \overrightarrow{f} \rangle.$$

Remark Note that $\langle \overrightarrow{f} \rangle_M$ depends on (f_0, c_0) and is only unique modulo $T_{c_0} I_{f_0}$.

2.2 Pencil condition

Recall that the pencil condition for $\mathbb{P}$ requires that a rational curve $c \in G_{\mathbb{P}}$ lies in one generic hypersurface $f \in \mathbb{P}$, but does not lie in any other hypersurfaces in the collection $\mathbb{P}$. The condition is necessary for Proposition 1.7. In the following lemma we give sufficient conditions for surfaces $\mathbb{P}$ to satisfy pencil condition.

Lemma 2.3. *Let $f_0 \in \mathbb{P}$ be S-generic. Let $V_i, i = 1, 2$ be irreducible subvarieties of S such that for each i, the intersection of all hypersurfaces in V_i is empty. Then for generic $(g_1, g_2) \in V_1 \times V_2$, the plane $\mathbb{P} = span(f_0, g_1, g_2)$ satisfies the pencil condition. In particular generic $\mathbb{P}$ in S satisfies the pencil condition.*

Proof. Since f_0 is S-generic, Definition 2.2 is valid around f_0. So Lemma 2.1 and Definition 2.2 yield

Claim 2.4. *for a fixed* $(f_0, c_0) \in \Gamma$ *(satisfying the surjectivity condition (2.2)),* $\langle \overrightarrow{f} \rangle_M \notin T_{c_0} I_{f_0}$ *if and only if the rational curve* $c_0(\mathbf{P}^1) \not\subset f$*, i.e if and only if* f *does not contain* c_0*.*

Then by the genericity of g_1, Claim 2.4 yields that

$$\langle \overrightarrow{g}_1 \rangle_M \notin T_{c_0} I_{f_0}$$

i.e.

$$T_{c_0} I_{span(f_0, g_1)} = T_{c_0} I_{f_0} \oplus \mathbb{C} \langle \overrightarrow{g}_1 \rangle_M. \tag{2.4}$$

Now we extend the *span*(f_0, g_1) by g_2. Suppose $\langle \overrightarrow{g}_2 \rangle_M$ lied in

$$T_{c_0} I_{f_0} \oplus \mathbb{C} \langle \overrightarrow{g}_1 \rangle_M.$$

Then

$$\langle \overrightarrow{ag_2 - bg_1} \rangle_M \in T_{c_0} I_{f_0} \tag{2.5}$$

where a, b are complex numbers. Then $ag_2 - bg_1$ would've contained the rational curve c_0. Furthermore g_2 would've contained the intersection points of g_1 and c_0. This is impossible because the intersection of all the hypersurfaces ag_2 is empty (if $a \neq 0$). Therefore

$$T_{c_0} I_{span(f_0, g_1, g_2)} = T_{c_0} I_{f_0} \oplus \mathbb{C} \langle \overrightarrow{g}_1 \rangle_M \oplus \mathbb{C} \langle \overrightarrow{g}_2 \rangle_M. \tag{2.6}$$

Suppose c_0 lied in some hypersurace $\epsilon_0 f_0 + \epsilon_1 g_1 + \epsilon_2 g_2$ where ϵ_1, ϵ_2 are non-zero complex numbers. Then it would've lied in $\epsilon_1 g_1 + \epsilon_2 g_2$. By claim 2.4,

$$\langle \overrightarrow{\epsilon_1 g_1 + \epsilon_2 g_2} \rangle_M$$

lied in $T_{c_0} I_{f_0}$. This contradicts (2.6). We complete the proof.

□

Pencil condition allows us to reduce the problem to the surjectivity of the differential $(\nu_1)_*$.

2.3 Zariski tangent spaces

A general result in the deformation theory reveals that the Hilbert scheme can be determined by the normal sheaf. So in this section we study how the normal sheaf relates the alternative of the Hilbert scheme.

Lemma 2.5. *Let f_0 be generic in S, and $\mathbb{L}_2 \subset S$ be the pencil spanned by f_0 and another hypersurface f_2. Assume they determine the components $I_{f_0}, I_{\mathbb{L}_2}$ satisfying*

$$I_{f_0} \subset I_{\mathbb{L}_2}, c_0^*(f_2) \neq 0 \textit{ with } c_0 \in I_{f_0} \tag{2.7}$$

Then

(a)

$$\frac{T_{c_0} I_{f_0}}{ker} \simeq H^0(c_0^*(T_{X_0})). \tag{2.8}$$

where ker is a line through the origin in $T_{c_0} I_{f_0}$ and $X_0 = div(f_0)$.

(b)

$$dim(T_{c_0} I_{\mathbb{L}_2})) = dim(T_{c_0} I_{f_0}) + 1, \tag{2.9}$$

Proof. (a). There is a regular map of the evaluation:

$$\begin{array}{rcl} e : M_d \times \mathbf{P}^1 & \to & \mathbf{P}^n \\ (c, t) & \to & c(t). \end{array} \tag{2.10}$$

Then its differential map point-wisely gives a rise to a homomorphism

$$\begin{array}{rcl} e_* : T_{c_0} M_d & \to & H^0(c_0^*(T_{\mathbf{P}^n})) \\ \alpha & \to & c_0^*(e_*(\alpha)). \end{array} \tag{2.11}$$

Let's analyze it. Let $M^0, \cdots, M^n$ be the subsets of $T_{c_0} M_d = M$, that are the $(n+1)$ tuples of $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ in M. Because c_0 is birational, through the rational projections of $\mathbf{P}^n$ to one of its $n+1$ coordinates components $z_0, z_1, \cdots, z_n$, we obtain the $n+1$ identity maps

$$e_*|_{M^i} : M^i \quad \to \quad H^0(c_0^*(\mathcal{O}_{\mathbf{P}^n}(1))) \simeq H^0(\mathcal{O}_{\mathbf{P}^1}(d))$$

for $i = 0, \cdots, n$. Then these maps give an isomorphism

$$M = T_{c_0} M_d \simeq H^0(c_0^*(\mathcal{O}_{\mathbf{P}^n}(1))^{\oplus(n+1)}). \tag{2.12}$$

Projectivizing both sides, we obtain the isomorphism

$$\varsigma' : T_{[c_0]}\mathbf{P}(M_d) \to H^0(c_0^*(T_{\mathbf{P}^n})). \tag{2.13}$$

The hypersurface $div(f_0) = X_0$ inserts the isomorphic subspaces to both sides of (2.13), defined by the vanishing of partial derivatives at c_0 of the coefficients of $f_0(c(t))$ (through the identification of (2.13)),

$$\begin{array}{ccc} T_{c_0}\mathbf{P}(M_d) & \simeq & H^0(c_0^*(T_{\mathbf{P}^n})) \\ \cup & & \cup \\ T_{[c_0]}\mathbf{P}(I_{f_0}) & \simeq & H^0(c_0^*(T_{X_0})) \end{array} \tag{2.14}$$

Notice that

$$T_{[c_0]}\mathbf{P}(I_{f_0}) = \frac{T_{c_0}I_{f_0}}{ker} \tag{2.15}$$

where ker is the equivalence line from the projectivization. This completes the proof of part (a).

(b). Denote the composition of

$$T_{c_0}M_d \to T_{[c_0]}\mathbf{P}(M_d) \xrightarrow{\varsigma'} H^0(c_0^*(T_{\mathbf{P}^n})) \tag{2.16}$$

by ς. Recall in Definition 2.2, we denote a non-zero vector in

$$\varsigma^{-1}(\langle \overrightarrow{f_2} \rangle) \tag{2.17}$$

by

$$\langle \overrightarrow{f_2} \rangle_M$$

Since

$$\frac{\partial f_0(c_0(t))}{\partial \langle \overrightarrow{f_2} \rangle} = -c_0^*(f_2) \neq 0, \tag{2.18}$$

(we regard $f(c(t))$ as a function in f, c) by part (a), $\langle \overrightarrow{f_2} \rangle_M$ does not lie in $T_{c_0}I_{f_0}$. Since

$$T_{c_0}I_{\mathbb{L}_2} = T_{c_0}I_{f_0} + \mathbb{C}\langle \overrightarrow{f_2} \rangle_M, \tag{2.19}$$

$T_{c_0}I_{\mathbb{L}_2}$ has dimension $dim(T_{c_0}I_{f_0}) + 1$.

□

Lemma 2.6. *Let f_0, f_1, f_2 be linearly independent in S and f_0 be S-generic. Assume $\mathbb{P} = span(f_0, f_1, f_2)$ satisfies the pencil condition. Recall*

$$\mathbb{L}_2 = span(f_0, f_2).$$

We choose components

$$I_{f_0} \subset I_{\mathbb{L}_2} \subset I_{\mathbb{P}},$$

and let $c_0 \in I_{f_0}$. Then

$$dim(T_{c_0} I_{\mathbb{P}}) = dim(T_{c_0} I_{\mathbb{L}_2}) + 1. \tag{2.20}$$

Furthermore

$$dim(T_{(f_0,c_0)} \Gamma_{\mathbb{P}}) = dim(T_{c_0} I_{\mathbb{P}}).$$

Proof. This is proved in Lemma 2.3 by the formula (2.6). □

Lemmas 2.3 -2.6 proved Proposition 1.4. In the rest of the section, we only concentrate on Proposition 1.5.

3 Calabi-Yau and Fano

Theorem 1.1 for the Calabi-Yau and Fano case follows from Proposition 1.5, which has been shown to be a consequence of Proposition 1.7. The idea in proving Proposition 1.7 is computational and we try to achieve a "simpler" representation of the Jacobian matrix through a search of a "better" Jacobian data. In this section, all neighborhoods and the word "local" are in the sense of Euclidean topology. It is divided into three steps. Each subsection contains one.

Subsection 3.1: We add 6 components to the direct ν_1 to obtain the indirect holomorphic map

$$\nu_2 : M \quad \rightarrow \quad \mathbb{C}^{hd+5} \tag{3.1}$$

The surjectivity of $(\nu_2)_*$ at a point on $I_{\mathbb{P}}$ implies the surjectivity of $(\nu_1)_*$ at the same point. The important realization in this step is that the surjectivity of $(\nu_2)_*$ can be mostly determined by the specialization of f_1, f_2 (which plays a role of the 1st order deformation of the rational curve). This allows us to have an accessible computation without a specialization of the geometry (which must involve f_0).

Subsection 3.2: Let $c_g \in M_d$ be a point. We'll construct local analytic coordinates of M_d around c_g, called quasi-polar coordinates. They will be used to analyze the Jacobian matrix of $(\nu_2)_*$.

Subsection 3.3: Adjust Jacobian data for ν_2, especially choose particular f_1, f_2. This allows us to break the representation matrix into block matrices, so we can deal blocks one-by-one after the specialization. Once $(\nu_2)_*$ is surjective for one special set of variables, we deform all variables to general positions.

3.1 Holomorphic maps

In this section we show the conversion from direct ν_1 to indirect ν_2.

Recall the definition of ν_1. First let $\mathbb{P}$ be a plane in S spanned by three hypersurfaces f_0, f_1, f_2. Choose $hd+1$ distinct, ordered designated points t_i on $\mathbb{C} \subset \mathbf{P}^1$, denoted by $\mathbf{t} = (t_1, \cdots, t_{hd+1})$. Then ν_1 is just the holomorphic map

$$\begin{aligned} \nu_1 : M &\to \mathbb{C}^{hd-1} \\ c &\to \left(\begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} \right)_{i=3,\cdots,hd+1} \end{aligned} \tag{3.2}$$

Expand the determinant in (3.2) along the first row

$$\begin{vmatrix} f_2(c(t_i)) & f_1(c(t_i)) & f_0(c(t_i)) \\ f_2(c(t_1)) & f_1(c(t_1)) & f_0(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix}$$

$$\|$$

$$\begin{vmatrix} f_1(c(t_1)) & f_0(c(t_1)) \\ f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} f_2(c(t_i)) + \begin{vmatrix} f_0(c(t_1)) & f_2(c(t_1)) \\ f_0(c(t_2)) & f_2(c(t_2)) \end{vmatrix} f_1(c(t_i))$$
$$+ \begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{vmatrix} f_0(c(t_i))$$

for $i=3,\cdots,hd+1$. Thus the differential $(\nu_1)_*$ has $hd-1$ coordinate's components of $\mathbb{C}^{hd-1}$,

$$\begin{aligned}\phi_i = & \left|\begin{array}{cc} f_1(c(t_1)) & f_0(c(t_1)) \\ f_1(c(t_2)) & f_0(c(t_2)) \end{array}\right| \mathbf{d}f_2(c(t_i)) + \left|\begin{array}{cc} f_0(c(t_1)) & f_2(c(t_1)) \\ f_0(c(t_2)) & f_2(c(t_2)) \end{array}\right| \mathbf{d}f_1(c(t_i)) \\ & + \left|\begin{array}{cc} f_2(c(t_1)) & f_1(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{array}\right| \mathbf{d}f_0(c(t_i)) + \sum_{l=0,j=1}^{l=2,j=2} h_{lj}^i(c)\mathbf{d}f_l(c(t_j))\end{aligned} \tag{3.3}$$

for $i=3,\cdots,hd+1$, where $\mathbf{d}$ is the differential.

Define three numbers for a fixed rational curve $c_g \in M_d$,

$$\delta_1 = \left|\begin{array}{cc} f_0(c_g(t_1)) & f_2(c_g(t_1)) \\ f_0(c_g(t_2)) & f_2(c_g(t_2)) \end{array}\right|,$$

$$\delta_2 = \left|\begin{array}{cc} f_1(c_g(t_1)) & f_0(c_g(t_1)) \\ f_1(c_g(t_2)) & f_0(c_g(t_2)) \end{array}\right| \tag{3.4}$$

$$\delta_0 = \left|\begin{array}{cc} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{array}\right|$$

Then define the hypersurface f_3 by

$$f_3 = \delta_2 f_2 + \delta_1 f_1 + \delta_0 f_0. \tag{3.5}$$

Proposition 3.1. *Let ν_2 be the holomorphic map*

$$\nu_2 : M \quad \rightarrow \quad \mathbb{C}^{hd+5} \tag{3.6}$$

given by $hd+5$ polynomials

$$\begin{array}{l} f_0(c(t_1)), f_0(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_2(c(t_1)), f_2(c(t_2)) \\ f_3(c(t_3)), f_3(c(t_4)), f_3(c(t_5)), \cdots, f_3(c(t_{hd})), f_3(c(t_{hd+1})). \end{array} \tag{3.7}$$

Then the surjectivity of $(\nu_2)_*$ at the point c_g implies the surjectivity of $(\nu_1)_*$ at the same point.

Remark If we choose variables t_1, t_2, c_g, f_1, f_2 satisfying one equation,

$$\left|\begin{array}{cc} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{array}\right| = 0, \tag{3.8}$$

i.e. $\delta_0 = 0$, then the surjectivity of $(\nu_2)_*$ at c_g, therefore the surjectivity of $(\nu_1)_*$ at c_g can be computed by the specialization of f_1, f_2.

.

Proof. The differential of ν_1 consists of $hd-1$ components

$$\phi_3, \cdots, \phi_{hd+1}$$

where each component is

$$\begin{aligned}\phi_i = &\begin{vmatrix} f_1(c(t_1)) & f_0(c(t_1)) \\ f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} \mathbf{d}f_2(c(t_i)) + \begin{vmatrix} f_0(c(t_1)) & f_2(c(t_1)) \\ f_0(c(t_2)) & f_2(c(t_2)) \end{vmatrix} \mathbf{d}f_1(c(t_i)) \\ &+ \begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{vmatrix} \mathbf{d}f_0(c(t_i)) + \sum_{l=0,j=1}^{l=2,j=2} h_{lj}^i(c)\mathbf{d}f_l(c(t_j)).\end{aligned} \tag{3.9}$$

If $(\nu_2)_*$ is surjective at the point c_g, $hd+5$ differential 1-forms,

$$\begin{array}{c}\mathbf{d}f_0(c(t_1)), \mathbf{d}f_0(c(t_2)), \mathbf{d}f_1(c(t_1)), \mathbf{d}f_1(c(t_2)), \mathbf{d}f_2(c(t_1)), \mathbf{d}f_2(c(t_2)) \\ \mathbf{d}f_3(c(t_3)), \mathbf{d}f_3(c(t_4)), \mathbf{d}f_3(c(t_5)), \cdots, \mathbf{d}f_3(c(t_{hd})), \mathbf{d}f_3(c(t_{hd+1})).\end{array} \tag{3.10}$$

when evaluated at c_g are linearly independent in the cotangent space $T^*_{c_g}M$. Thus the particular expression of formula (3.9) shows that the differential 1-forms

$$\phi_3, \cdots, \phi_{hd+1}$$

are also linearly independent in the same cotangent space $T^*_{c_g}M$. Hence ν_1 is surjective at the same point.

□

3.2 Quasi-polar coordinates

We introduce local analytic coordinates of the affine space M, that will simplify expressions of differentials on M.

Definition 3.2. *(polar coordinates for polynomials of one variable) Let $a_0 \in H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ be a non-zero element satisfying that the zeros are distinct. Then there is a Euclidean neighborhood $U \subset H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ of a_0, which has analytic coordinates $r, w_1, \cdots, w_d$ ($r \neq 0$) such that any element $a \in U$ has an expression*

$$a = r\prod_{j=1}^{d}(t - w_j). \tag{3.11}$$

We call $\{r, w_1, \cdots, w_d\}$ the polar coordinates of $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ at a_0.

Next we fix the notations of polar coordinates for each component $H^0(\mathcal{O}_{\mathbf{P}^1}(d))$ of M_d. Let non-zero $c=(c^0,\cdots,c^n)$ with

$$c^i \in H^0(\mathcal{O}_{\mathbf{P}^1}(d)), i=0,\cdots,n$$

be a varied point of M_d in a small analytic neighborhood centered around some point $c_g=(c_g^0,\cdots,c_g^n)$. We assume the equations $c^i(t)=0, i\leq n$, (including $c_g^i(t)=0$) always have hd distinct zeros

$$\theta_j^i, for\ 0\leq\ i\leq n, 1\leq j\leq d$$

Then we have polar coordinates for M around c_g. We denote them by

$$\begin{gathered} r_i, \theta_j^i, \\ j=1,\cdots,d, i=0,\cdots,n \end{gathered} \tag{3.12}$$

with $r_i\neq 0$ satisfying

$$c^i(t)=r_i\prod_{j=1}^{d}(t-\theta_j^i). \tag{3.13}$$

The values of the center point c_g of the neighborhood are denoted by

$$\begin{gathered} \mathring{r}_i, \mathring{\theta}_j^i \\ for\ i=0,\cdots,n, j=1,\cdots,d. \end{gathered}$$

Next we define quasi-polar coordinates that are associated to the special type of hypersurfaces we are going to choose later. They are partial polar coordinates for M_d with a replacement of last two components c^{n-1}, c^n. Let q be a homogeneous quadratic polynomial in variables $z_0,\cdots,z_n$. Let

$$h(c,t)=\delta_1 q(c(t))+\delta_2 c^{n-1}(t)c^n(t). \tag{3.14}$$

for $c\in M$, where $\delta_i, i=1,2$ are two complex numbers, generic in $\mathbb{C}^2$. Assume for c in a small analytic neighborhood, $h(c,t)=0$ has $2d$ distinct zeros. Let $\gamma_1,\cdots,\gamma_{2d}$ be the zeros of $h(c,t)=0$. Similar to the polar coordinates, we let

$$h(c,t)=R\prod_{k=1}^{2d}(t-\gamma_k), R\neq 0$$

It is clear that

$$R = \delta_1 q(r_0, r_1, r_2, r_3, r_4) + \delta_2 r_{n-1} r_n, \ \textit{and}$$
$$\gamma_k \ \textit{are analytic functions of } c.$$

(Notice R is the value of $h(c,t)$ at $t = \infty$, the coefficient of the highest order.). Let the coordinates values at the center point be $\mathring{R}, \mathring{\gamma}_k$.

Proposition 3.3. *Let δ_1, δ_2 and q be generic. Let $U_{c_g} \subset M$ be an analytic neighborhood of a center point c_g as above.*

Let

$$\varrho : U_{c_g} \quad \rightarrow \quad \mathbb{C}^{(n+1)(d+1)} \tag{3.15}$$

be a regular map that is defined by

$$\begin{array}{c} (\theta_1^0, \cdots, \theta_d^n, r_0, \cdots, r_n) \\ \downarrow \varrho \\ (\theta_1^0, \cdots, \theta_d^{n-2}, r_0, \cdots, r_n, \gamma_1, \cdots, \gamma_{2d}). \end{array} \tag{3.16}$$

Then ϱ is an isomorphism to its image.

Proof. It suffices to prove the complex differential of ϱ at c_g is an isomorphism for specific q, δ_i. So we assume that

$$\delta_1 = 0, \delta_2 = 1.$$

Then $h(c,t) = c^{n-1}(t)c^n(t)$. Hence $\gamma_k, k = 1, \cdots, 2d$ are just

$$\theta_j^i, i = n-1, n, j = 1, \cdots, d.$$

So ϱ is the identity map. We complete the proof.

□

Definition 3.4. *By Proposition 3.3, for generic δ_1, δ_2, q,*

$$\theta_1^0, \cdots, \theta_d^{n-2}, r_0, r_1, \cdots, r_n, \gamma_1, \cdots, \gamma_{2d} \tag{3.17}$$

are local analytic coordinates of M_d around c_g. We denote the system of coordinates by

$$C_M$$

and will be called quasi-polar coordinates.

Let's apply the quasi-polar coordinates to calculate a Jacobian matrix.

Choose a generic homogeneous coordinates $[z_0, \cdots, z_n]$ for $\mathbf{P}^n$. Let

$$f_3 = z_0 z_1 \cdots z_{n-2}(\delta_1 q + \delta_2 z_{n-1} z_n). \tag{3.18}$$

be a polynomial, where δ_1, δ_2, q are generic. Let $c_g \in M_d$ such that

$$f_3(c_g(t)) \neq 0.$$

Let C_M be the associated quasi-polar coordinates around c_g. We denote the zeros of $f_3(c_g(t)) = 0$ by

$$\tilde{t}_1, \tilde{t}_2, \cdots, \tilde{t}_{hd}.$$

Lemma 3.5. *Then*

(a) the Jacobian matrix

$$J(c_g) = \frac{\partial(f_3(c_g(\tilde{t}_1)), \cdots, f_3(c_g(\tilde{t}_{hd}))}{\partial(\theta_1^0, \cdots, \theta_d^{n-2}, \gamma_1, \cdots, \gamma_{2d})} \tag{3.19}$$

is equal to a diagonal matrix D whose diagonal entries are non-zero partial derivatives with respect to the variable c evaluated at c_g,

$$\frac{\partial f_3(c_g(\tilde{t}_1))}{\partial \theta_1^0}, \cdots, \frac{\partial f_3(c_g(\tilde{t}_{(h-2)d}))}{\partial \theta_d^{n-2}}, \frac{\partial f_3(c_g(\tilde{t}_{(h-2)d+1}))}{\partial \gamma_1}, \cdots, \frac{\partial f_3(c_g(\tilde{t}_{hd}))}{\partial \gamma_{2d}} \tag{3.20}$$

where $\theta^{\bullet}_{\bullet}, \gamma_{\bullet}$ from the C_M coordinates of M_d.

(b) For $i = 1, \cdots, hd$, $l = 0, \cdots, n$, the partial derivatives evaluated at c_g,

$$\frac{\partial f_3(c_g(\tilde{t}_i))}{\partial r_l} = 0.$$

Proof. Note $\mathring{\theta}_j^i, i = 0, \cdots, n, j = 1, \cdots, d$ and $\mathring{\gamma}_k, k = 0, \cdots, 2d$ are distinct. Thus the coordinates in Definition 3.4 exist. Using the C_M coordinates for the function $f_3(c(\tilde{t}_i))$ (of variable c), we have

$$f_3(c(t)) = r_0 \cdots r_{n-2} R \prod_{i=0, j=1, k=1}^{i=n-2, j=d, k=2d} (t - \theta_j^i)(t - \gamma_k). \tag{3.21}$$

Notice right hand side of (3.21) is in analytic coordinates C_M, and R is a polynomial in variables $r_0, \cdots, r_n$. Both parts of Lemma 3.5 follow from the expression (3.21). We complete the proof. $\square$

3.3 Specialization of Jacobian data

Now we are ready to make the computation for Proposition 1.7. It has two steps.

1st step: Specialization of Jacobian data. There are 4 types of variables in Jacobian data to be specialized and adjusted: intrinsic hypersurfaces f_1, f_2, rational curve c_g, and extrinsic coordinates z_i, designated points $t_i \in \mathbf{P}^1$.

Let $z_0, \cdots, z_n$ be general homogeneous coordinates of $\mathbf{P}^n$. Let f_0 be S-generic. Let

$$f_2 = z_0 z_1 \cdots z_n,$$
$$f_1 = z_0 \cdots z_{n-2} q,$$

where q is a generic quadratic homogeneous polynomial in $z_0, \cdots, z_n$.

We'll work with the special plane $\mathbb{P}$ spanned by f_0, f_1, f_2, which by Lemma 2.3 satisfies the pencil condition. Now we continue the selection of Jacobian data. Let

$$c_g \in I_{\mathbb{P}}$$

be a generic point such that

$$c_g = (c_g^0, \cdots, c_g^n)$$

satisfies that $c_g^i \neq 0$ for all i and equations

$$c_g^i(t) = 0, i = 0, \cdots, n$$

have $(n+1)d$ distinct zeros $\mathring{\theta}_j^i \in \mathbf{P}^1$. (It is quite important to notice that c_g does not lie in any individual f_0, f_1, f_2, but it does lie in a unspecified linear combination of them). Let's choose special $hd+1$ distinct points t_i on $\mathbb{C} \subset \mathbf{P}^1$, denoted by $\mathbf{t}_s = (t_1, \cdots, t_{5d+1})$ (the designated points).

(1) t_{hd+1} is generic and variables t_1, t_2, z_i, q, c_g satisfy one equation

$$\begin{vmatrix} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{vmatrix} = 0, \tag{3.22}$$

(this is the choice for t_{hd+1}, t_1, t_2)

(2) $t_3, \cdots, t_{hd}$ are the $hd-2$ complex numbers

$$\mathring{\theta}_j^i, \mathring{\gamma}_k, \quad (i,j) \neq (0,1), (1,1)$$

$$1 \leq k \leq 2d, 0 \leq i \leq n-3, 1 \leq j \leq d.$$

that are all zeros of

$$f_3(c_g(t)) = \delta_1 f_1(c_g(t)) + \delta_2 f_2(c_g(t)) = 0. \tag{3.23}$$

but excluding two zeros $\mathring{\theta}_1^0, \mathring{\theta}_1^1$, where

$$\begin{aligned} \delta_1 &= \begin{vmatrix} f_0(c_g(t_1)) & f_2(c_g(t_1)) \\ f_0(c_g(t_2)) & f_2(c_g(t_2)) \end{vmatrix}, \\ \delta_2 &= \begin{vmatrix} f_1(c_g(t_1)) & f_0(c_g(t_1)) \\ f_1(c_g(t_2)) & f_0(c_g(t_2)) \end{vmatrix}. \end{aligned} \tag{3.24}$$

Let's see δ_1, δ_2 are distinct. By the pencil condition,

$$\begin{aligned} &\Big(f_2(c(t_1)), f_1(c(t_1)), f_0(c(t_1))\Big) \\ &\Big(f_2(c(t_2)), f_1(c(t_2)), f_0(c(t_2))\Big). \end{aligned} \tag{3.25}$$

span a 2 dimensional plane. We can choose t_1, t_2, z_i, q, c_g such that

$$\begin{vmatrix} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{vmatrix} = 0 \tag{3.26}$$

and also allow the complex numbers δ_1, δ_2 to be generic due to the genericity of q (for instance, a generic constant multiple of q will give the genericity of (δ_1, δ_2)).

Using these selected Jacobian data, let's recall the formulation of differential algebra. Applying designated points $t_1, \cdots, t_{hd+1}$ we obtain the differential of ν_1, whose each component is

$$\begin{aligned} \phi_i = &\begin{vmatrix} f_1(c(t_1)) & f_0(c(t_1)) \\ f_1(c(t_2)) & f_0(c(t_2)) \end{vmatrix} \mathbf{d} f_2(c(t_i)) + \begin{vmatrix} f_0(c(t_1)) & f_2(c(t_1)) \\ f_0(c(t_2)) & f_2(c(t_2)) \end{vmatrix} \mathbf{d} f_1(c(t_i)) \\ &+ \begin{vmatrix} f_2(c(t_1)) & f_1(c(t_1)) \\ f_2(c(t_2)) & f_1(c(t_2)) \end{vmatrix} \mathbf{d} f_0(c(t_i)) + \textstyle\sum_{l=0,j=1}^{l=2,j=2} h_{lj}^i(c_g) \mathbf{d} f_l(c(t_j)) \end{aligned} \tag{3.27}$$

for $i = 3, \cdots, hd+1$. Let's evaluated at c_g. By the only constraint (3.22),

$$\begin{vmatrix} f_2(c_g(t_1)) & f_1(c_g(t_1)) \\ f_2(c_g(t_2)) & f_1(c_g(t_2)) \end{vmatrix} = 0.$$

Then we obtain

$$\begin{aligned} \phi_i|_{c_g} &= \delta_1 \mathbf{d} f_1(c(t_i))|_{c_g} + \delta_2 \mathbf{d} f_2(c(t_i))|_{c_g} + \textstyle\sum_{l=0,j=1}^{l=2,j=2} h_{lj}^i(c_g) \mathbf{d} f_l(c(t_j))|_{c_g} \\ &= \mathbf{d} f_3(c(t_i))|_{c_g} + \textstyle\sum_{l=0,j=1}^{l=2,j=2} h_{lj}^i(c_g) \mathbf{d} f_l(c(t_j))|_{c_g} \end{aligned} \tag{3.28}$$

where

$$f_3 = \delta_1 f_1 + \delta_2 f_2. \tag{3.29}$$

Notice (δ_1, δ_2) is generic.

Applying Proposition 3.1, we switch the ν_1 to the differential of another holomorphic map ν_2.

Claim 3.6. *$(\nu_2)_*$ is surjective at a generic point $c_g \in I_{\mathbb{P}}$.*

Proof. of Claim 3.6: The Jacobian matrix for ν_2 is not a square matrix. To have a square matrix, we select a square minor in the Jacobian matrix of ν_1 in the following way. We may assume $h \geq 2$. First we choose the smooth subvariety M_s in the analytic neighborhood of M that is defined by

$$\begin{cases} r_3 = \cdots = r_{n-2} = 0 \\ \theta_i^j = 0, i = 1, \cdots, d, j = h-2, \cdots, n-2. \end{cases}$$

So the non-zero C_M coordinates for M_s can be written

$$\theta_i^j, \gamma_1, \cdots \gamma_{2d}, r_0, r_1, r_2, r_{n-1}, r_n$$

where $i = 1, \cdots, d, j = 0, \cdots, h-3$. So there are $hd + 5$ analytic variables for local Euclidean space $M_s \simeq \mathbb{C}^{hd+5}$. (The requirement for this choice of indexes is $n \geq 4$). Let

$$\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t}) \tag{3.30}$$

be the Jacobian matrix of the restriction of ν_2 to M_s at c_g, under an analytic coordinates system C_M on M_s We break

$$\mathcal{A}(C_M, f_0, f_1, f_2, \mathbf{t})$$

to block matrices.

$$\begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{21} & \mathcal{A}_{22} \end{pmatrix} \tag{3.31}$$

where $\mathcal{A}_{ij}$ are the Jacobian matrices as follow:

(a)

$$\mathcal{A}_{11} = \frac{\partial(f_3(c(t_3)), f_3(c(t_4)), \cdots, f_3(c(t_{hd})))}{\partial(\theta_2^0, \cdots, \hat{\theta}_1^1, \cdots, \theta_{h-3}^d, \gamma_1, \cdots, \gamma_{2d})}, ((\hat{\cdot}) := omit) \tag{3.32}$$

(b)

$$\mathcal{A}_{12} = \frac{\partial(f_3(c(t_3)), f_3(c(t_4)), \cdots, f_3(c(t_{hd})))}{\partial(\theta_1^0, \theta_1^1, r_0, r_1, r_2, r_{n-1}, R)}. \tag{3.33}$$

(c)

$$\mathcal{A}_{21} = \frac{\partial(f_3(c(t_{hd+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2)))}{\partial(\theta_2^0, \cdots, \hat{\theta}_1^1, \cdots, \theta_{h-3}^d, \gamma_1, \cdots, \gamma_{2d})}. \tag{3.34}$$

(d)

$$\mathcal{A}_{22} = \frac{\partial(f_3(c(t_{hd+1})), f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2)))}{\partial(\theta_1^0, \theta_1^1, r_0, r_1, r_2, r_{n-1}, R)}. \tag{3.35}$$

Using Lemma 3.5, $\mathcal{A}_{11}|_{c_g}$ is a non-zero diagonal matrix and

$$\mathcal{A}_{12}|_{c_g} = 0.$$

Therefore it suffices to show

$$det(\mathcal{A}_{22})|_{c_g} \neq 0. \tag{3.36}$$

Notice t_{hd+1} is generic on $\mathbf{P}^1$. The genericity of q makes curve in $\mathbb{C}^7$,

$$\left(\frac{\partial f_3(c(t))}{\partial \theta_1^0}, \frac{\partial f_3(c(t))}{\partial \theta_1^1}, \frac{\partial f_3(c(t))}{\partial r_0}, \cdots, \frac{\partial f_3(c(t))}{\partial r_n}\right)|_{c_g} \tag{3.37}$$

span the entire space $\mathbb{C}^7$. This means the first row vector of

$$\mathcal{A}_{22}(C_M)|_{c_g}$$

which varies with t_{hd+1} is generic with respect to other 6 row vectors. Hence it suffices for us to show the Jacobian matrix

$$\mathcal{B}(c_g) = \left.\frac{\partial(f_2(c(t_1)), f_2(c(t_2)), f_1(c(t_1)), f_1(c(t_2)), f_0(c(t_1)), f_0(c(t_2)))}{\partial(\theta_1^0, \theta_1^1, r_1, r_2, r_{n-1}, r_n)}\right|_{c_g} \tag{3.38}$$

is non degenerate (the column of partial derivatives with respect to r_0 is eliminated). To do that, it suffices to show it is non-degenerate for a special $c'_g \in I_{\mathbb{P}}$. So we let $\mathbb{L}_2$ be the pencil through f_0, f_2. There is a component $I_{\mathbb{P}}$ containing the component $I_{\mathbb{L}_2}$ where q from f_2 is generic. Let c'_g be a generic point of $I_{\mathbb{L}_2}$ (c'_g lies in a lower dimensional subvariety $I_{\mathbb{L}_2}$, but it is still in M_d because f_0 is S-generic.). Because q is generic with respect to 1st, 2nd, 5th and 6th rows, two middle rows of the matrix $\mathcal{B}(c_g)$,

$$\begin{array}{c} (\frac{\partial f_1(c(t_1))}{\partial \theta_1^0}, \frac{\partial f_1(c(t_1))}{\partial \theta_1^1}, \frac{\partial f_1(c(t_1))}{\partial r_1}, \cdots, \frac{\partial f_1(c(t_1))}{\partial r_n})|_{c'_g} \\ (\frac{\partial f_1(c(t_2))}{\partial \theta_1^0}, \frac{\partial f_1(c(t_2))}{\partial \theta_1^1}, \frac{\partial f_1(c(t_2))}{\partial r_1}, \cdots, \frac{\partial f_1(c(t_2))}{\partial r_n})|_{c'_g} \end{array} \tag{3.39}$$

in $\mathbb{C}^6$ must be linearly independent of 1st, 2nd, 5th and 6th rows (because q can vary freely as c'_g stays fixed). Then we reduce the non-degeneracy of $\mathcal{B}(c'_g)$ to the non-degeneracy of 4×4 matrix

$$Jac(f_0, c'_g) = \frac{\partial\big(f_2(c(t_1)), f_2(c(t_2)), f_0(c(t_1)), f_0(c(t_2))\big)}{\partial(\theta_1^0, r_2, r_{n-1}, r_n)}|_{c'_g}. \tag{3.40}$$

Finally we write down the matrix $Jac(f_0, c'_g)$,

$$\begin{array}{c} Jac(f_0, c'_g) \\ \| \\ \lambda \begin{pmatrix} \frac{1}{t_1-\mathring{\theta}_1^0} & 1 & 1 & 1 \\ \frac{1}{t_2-\mathring{\theta}_1^0} & 1 & 1 & 1 \\ \frac{\partial f_0(c'_g(t_1))}{\partial \theta_1^0} & (z_2\frac{\partial f_0}{\partial z_2})|_{c'_g(t_1)} & (z_{n-1}\frac{\partial f_0}{\partial z_{n-1}})|_{c'_g(t_1)} & (z_n\frac{\partial f_0}{\partial z_n})|_{c'_g(t_1)} \\ \frac{\partial f_0(c'_g(t_2))}{\partial \theta_1^0} & (z_2\frac{\partial f_0}{\partial z_2})|_{c'_g(t_2)} & (z_{n-1}\frac{\partial f_0}{\partial z_{n-1}})|_{c'_g(t_2)} & (z_n\frac{\partial f_0}{\partial z_n})|_{c'_g(t_2)} \end{pmatrix}, \end{array} \tag{3.41}$$

where λ is a non-zero complex number. We further compute to have

$$\begin{array}{c} Jac(f_0, c'_g) \\ \| \\ \lambda(\frac{1}{t_1-\mathring{\theta}_1^0} - \frac{1}{t_2-\mathring{\theta}_1^0}) \begin{pmatrix} 1 & 1 & 1 \\ (z_2\frac{\partial f_0}{\partial z_2})|_{c'_g(t_1)} & (z_{n-1}\frac{\partial f_0}{\partial z_{n-1}})|_{c'_g(t_1)} & (z_n\frac{\partial f_0}{\partial z_n})|_{c'_g(t_1)} \\ (z_2\frac{\partial f_0}{\partial z_2})|_{c'_g(t_2)} & (z_{n-1}\frac{\partial f_0}{\partial z_{n-1}})|_{c'_g(t_2)} & (z_n\frac{\partial f_0}{\partial z_n})|_{c'_g(t_2)} \end{pmatrix}, \end{array} \tag{3.42}$$

where $\mathring{\theta}_1^0$ is a complex number. Since all the variables t_1, t_2, z_i, q are only required to satisfy one equation (3.22), we may assume $(t_1, t_2) \in \mathbb{C}^2$ is generic. Let's now prove the non-degeneracy of $Jac(f_0, c'_g)$. First we identify the hypersurface containing c'_g as f'. Then we consider the Jacobian,

$$J(f', c'_g) = \begin{vmatrix} 1 & 1 & 1 \\ (z_2\frac{\partial f'}{\partial z_2})|_{c'_g(t_1)} & (z_{n-1}\frac{\partial f'}{\partial z_{n-1}})|_{c'_g(t_1)} & (z_n\frac{\partial f'}{\partial z_n})|_{c'_g(t_1)} \\ (z_2\frac{\partial f'}{\partial z_2})|_{c'_g(t_2)} & (z_{n-1}\frac{\partial f'}{\partial z_{n-1}})|_{c'_g(t_2)} & (z_n\frac{\partial f'}{\partial z_n})|_{c'_g(t_2)} \end{vmatrix}.$$

We calculate

$$J(f', c'_g) = \begin{vmatrix} f_4(c'_g(t_1)) & f_5(c'_g(t_1)) \\ f_4(c'_g(t_2)) & f_5(c'_g(t_2)) \end{vmatrix}.$$

where

$$f_4 = z_2\frac{\partial f'}{\partial z_2} - z_n\frac{\partial f'}{\partial z_n}$$
$$f_5 = z_{n-1}\frac{\partial f'}{\partial z_{n-1}} - z_n\frac{\partial f'}{\partial z_n}$$

are two hypersurfaces. If $J(f', c'_g) = 0$, then by the genericity of t_1, t_2, the two dimensional vectors vectors

$$\left(f_4(c'_g(t)), f_5((c'_g(t)\right), \ all\ t$$

must span a line. So there exist two complex numbers ϵ_1, ϵ_2 not all zeros such that

$$(\epsilon_1 z_2\frac{\partial f'}{\partial z_2} + \epsilon_2 z_{n-1}\frac{\partial f'}{\partial z_{n-1}} + (-\epsilon_1 - \epsilon_2)z_n\frac{\partial f'}{\partial z_n})|_{c'_g(t)} = 0. \tag{3.43}$$

Then the t-varied vector

$$\eta = (0, 0, \epsilon_1 c'^2_g, 0, \cdots, 0, \epsilon_2 c'^{n-1}_g, (-\epsilon_1 - \epsilon_2)c'^n_g)$$

is the non-zero holomorphic section of $(c'_g)^*(T_{X'})$, where $div(f') = X'$ is the generic and c'^i_g is the i-th component of c'_g. Notice that

$$(c'_g)^*(T_{X'}) \tag{3.44}$$

has rank $n-1$. Using a generic coordinates z_i, the pullback of the plane $\{z_0 = z_1 = z_3 = \cdots = z_{n-2} = 0\}$ to the bundle $c_0^*(\mathbf{P}^n)$ defines a rank 1 subbundle E of $(c'_g)^*(T_{X'})$, where the η lies in. On the other hand the tangent vector of the rational curve $c'_g(\mathbf{P}^1)$ at generic points should also be cut (by the above plane) into this rank 1 bundle. Hence η must be parallel to the rational curve after mod-out $\{z_0 = z_1 = z_3 = \cdots = z_{n-2} = 0\}$. This is impossible by the genericity of z_i coordinates. We complete the proof of Claim 3.6.

2nd step: Let's deform to a general position. By the claim 3.6, we deform $\mathbb{P}$ to the general position. Hence we proved Proposition 1.7 at generic points of $I_{\mathbb{P}}$ for a generic $\mathbb{P}$. Suppose c_g is not generic, i.e. $c_g \in G_{\mathbb{P}}$ satisfies that

$$(\nu_1)_*|_c : T_c I_{\mathbb{P}} \ \ \rightarrow \ \ T_0\mathbb{C}^{hd-1} \tag{3.45}$$

is not surjective. Then by the genericity of the hypersurface associated to the rational curve c_g, there is an irreducible subvariety

$$\Sigma_{\mathbb{P}} \subset \Gamma_{\mathbb{P}}$$

dominating $\mathbb{P}$ such that for generic $(f, c_g) \in \Sigma_{\mathbb{P}}$,

$$(\nu_2)_*|_{c_g} : T_{c_g} I_{\mathbb{P}} \quad \rightarrow \quad T_0 \mathbb{C}^{hd-1} \tag{3.46}$$

is not surjective. where $I_{\mathbb{P}}$ is the component containing $P_l(\Sigma_{\mathbb{P}})$. On the other hand, by the dominance of $\Sigma_{\mathbb{P}} \rightarrow \mathbb{P}$, the proof for claim 3.6 will hold. It shows that at generic point c_g of $P_l(\Sigma_{\mathbb{P}})$, $(\nu_1)_*|_{c_g}$ is surjective. This contradicts the choice of $\Sigma_{\mathbb{P}}$, which says that $(\nu_2)_*|_{c_g}$ is not surjective. Therefore Proposition 1.7 holds at all points $c_g \in G_{\mathbb{P}}$. □

3.4 Hilbert scheme $\mathcal{M}_d(X)$

In this subsection we come back to the invariant Hilbert scheme to prove Theorem 1.1. We'll show the results in Propositions 1.4, 1.5, 1.7 for the alternative lead to the consequences of the normal sheaf, then to the Hilbert scheme.

Proof. of Theorem 1.1: Now we give the proof Theorem 1.1, which changes the focus from the alternative to the Hilbert scheme. Let $\mathbb{P} = span(f_0, f_1, f_2)$ be generic, and $c_0 \in G_{\mathbb{P}}$. We may allow $c_0(\mathbf{P}^1) \subset f_0$. By Proposition 1.5,

$$(n+1-h)d + n + 2 \tag{3.47}$$

is the dimension of the Zariski tangent space $T_{c_0} I_{\mathbb{P}}$. Furthermore using Lemma 2.5 and Lemma 2.6, we obtain that

$$dim(T_{c_0} I_{f_0}) = (n+1-h)d + n. \tag{3.48}$$

Then we obtain that

$$dim(H^0(N_{c_0/X})) = (n+1-h)d + n - 4. \tag{3.49}$$

Now we consider the exact sequence of sheaf modules on $\mathbf{P}^1$,

$$0 \quad \rightarrow \quad N_{c_0/X} \quad \rightarrow \quad N_{c_0/\mathbf{P}^n} \quad \rightarrow \quad c_0^*(N_{X/\mathbf{P}^n}) \quad \rightarrow \quad 0. \tag{3.50}$$

This induces the exact sequence of finite dimensional linear spaces

$$0 \rightarrow H^0(N_{c_0/X}) \rightarrow H^0(N_{c_0/\mathbf{P}^n}) \rightarrow H^0(c_0^*(N_{X/\mathbf{P}^n})) \rightarrow H^1(N_{c_0/X}) \rightarrow 0.$$

This implies that

$$dimH^1(N_{c_0/X}) = dimH^0(N_{c_0/\mathbf{P}^n}) - dimH^0(N_{c_0/X}) - dimH^0(c_0^*(N_{X/\mathbf{P}^n})). \tag{3.51}$$

Using Euler sequence for $\mathbf{P}^n$, we obtain

$$dim(H^0(N_{c_0/\mathbf{P}^n})) = (n+1)(d+1) - 4. \tag{3.52}$$

Using adjunction formula, we obtain

$$dim(H^0(c_0^*(N_{X/\mathbf{P}^n})) = hd + 1. \tag{3.53}$$

Then substituing all terms in (3.51), we obtain

$$dim(H^1(N_{c_0/X})) = 0. \tag{3.54}$$

This shows the obstruction space to the deformation of the rational curve is zero. Applying the standard deformation technique as in Theorem 2.10, I, [3], we obtain that the local dimension of the Hilbert scheme $dim(\mathcal{M}_d(X)|_{C_0})$ at $C_0 = c_0(\mathbf{P}^1)$ has dimension at least

$$dim(H^0(N_{c_0/X})),$$

which is the dimension of the Zariski tangent space,

$$dim(T_{C_0}\mathcal{M}_d(X)).$$

That is

$$dim(\mathcal{M}_d(X)|_{C_0}) \geq dim(H^0(N_{c_0/X})).$$

On the other hand the scheme $\mathcal{M}_d(X)$ should always satisfy

$$dim(\mathcal{M}_d(X)|_{C_0}) \leq dim(T_{C_0}\mathcal{M}_d(X)) = dim(H^0(N_{c_0/X})).$$

Therefore

$$dim(T_{C_0}\mathcal{M}_d(X)) = dim(\mathcal{M}_d(X)|_{C_0})$$

is equal to $dim(H^0(N_{c_0/X}))$. Theorem 1.1 in the Calabi-Yau case at generic points is proved.

Finally we extend the result to all $c_g \in I_{\mathbb{L}}$ (non-generic points). To see this, we suppose there is a birational-to-its-image map c_g for each generic X such that $H^1(N_{c_g/X}) \neq 0$. Then there is a subvariety $\Theta \subset \Gamma$ DOMINATING S such that for all $(c_g, f) \in \Theta$,

$$H^1(N_{c_g/X}) \neq 0.$$

Then we can repeat the same process to obtain that $H^1(N_{c_g/div(f_g)}) = 0$. This contradiction shows such Θ does not exist. (actually the only condition for the vanishing H^1 is that the component of the incidence scheme dominates S). This completes the proof of Theorem 1.1 for Calabi-Yau and Fano.

□

4 Hypersurfaces of general type

4.1 The case of $n \geq 4$

In this section, we prove Theorem 1.1 for hypersurfaces of general type, i.e. the case $n + 1 - h < 0$. This will follow from the Calabi-Yau case. We let

$$n + 1 + \delta = h \tag{4.1}$$

where integer $\delta \geq 1$.

Let

$$\nu : \mathbf{P}^{n+\delta} \dashrightarrow \mathbf{P}^n \tag{4.2}$$

be the projection from the infinity $\mathbf{P}^{\delta-1}$. At a point

$$a \in \mathbf{P}^{n+\delta} \backslash \mathbf{P}^{\delta-1}$$

the differential map

$$\nu_* : T_a\mathbf{P}^{n+\delta} \rightarrow T_{\nu(a)}P^n \tag{4.3}$$

is surjective. Let $F_0 \in H^0(\mathcal{O}_{\mathbf{P}^{n+\delta}}(h))$ be generic. F_0 is restricted to

$$f_0 \in H^0(\mathcal{O}_{\mathbf{P}^n}(h))$$

which is also generic. Let $c_0 \subset f_0$ be a rational curve in $\mathbf{P}^n$. We denote its inclusion in $\mathbf{P}^{n+\delta}$ by c_0^δ. By the projection (4.3),

$$\nu_*(T_{div(F_0)}) = \nu_*(T_{div(f_0)}). \tag{4.4}$$

Then we have an exact sequence of sheaves

$$0 \rightarrow K \rightarrow N_{c_0^\delta/div(F_0)} \rightarrow N_{c_0/div(f_0)} \rightarrow 0. \tag{4.5}$$

where K is the kernel. Notice all sheaves are over $\mathbf{P}^1$.

Therefore we have the exact sequence of vector spaces

$$H^1(N_{c_0^\delta/div(F_0)}) \rightarrow H^1(N_{c_0/div(f_0)}) \rightarrow H^2((c_0^\delta)^*(K)) = 0. \tag{4.6}$$

By Theorem 1.1 for the Calabi-Yau case,

$$H^1(N_{c_0^\delta/div(F_0)}) = 0.$$

Hence

$$H^1(N_{c_0/div(f_0)}) = 0.$$

Then we repeat Kollár's theorem 2.10, I, [2] as above. This completes the first part of Theorem 1.1 for hypersurfaces of general type at generic points.

Then by the same argument for the Calabi-Yau as above, we extend the result to all $c_g \in I_{\mathbb{L}}$. This completes the proof of Theorem 1.1.

4.2 The case of $n = 3$

We fill in the missing part in the proof of Corollary 1.2 for the case $n \leq 3$. This is not covered by Theorem 1.1. The case of $n = 2$ is classically known. So it suffices to prove it for the case $n = 3$.

Proof. We'll prove that a generic hypersurface in $\mathbf{P}^3$ of degree ≥ 5 does not admit irreducible rational curves of any degrees. [2] We prove it by a contradiction. Let X be a generic hypersurface of degree $h \geq 5$ in $\mathbf{P}^3$, defined by the polynomial f. Assume C is the rational curve of degree d on X and $c : \mathbf{P}^1 \rightarrow C$ is its normalization. Let g be a homogeneous linear polynomial of $\mathbf{P}^3$, defining a generic hyperplane. Let $l_1, \cdots, l_h$ be another

[2] The same statement for the case of immersed rational curves was proved in [2].

h homogeneous linear polynomials defining hyperplanes such that c does lie on them, and the equations

$$l_k(c(t)) = 0 = l(c(t)),\ for\ all\ k$$

have distinct roots at smooth locus of c. Because f is S-generic, we can use Definition 2.2 to obtain two sections of the bundle $c^*(T_{\mathbf{P}^3})$,

$$\begin{cases} \langle\overrightarrow{l_1 \cdots l_h}\rangle, \\ \langle\overrightarrow{l_1 \cdots g \cdots l_h}\rangle_i, i = 1, \cdots, h. \end{cases} \tag{4.7}$$

where $\langle\overrightarrow{l_1 \cdots g \cdots l_h}\rangle_i$ is a section of $c^*(T_{\mathbf{P}^3})$ corresponding to the hypersurface $l_1 \cdots g \cdots l_h$ with the substitution g at $i - th$ hyperplane l_i. Then we define

$$\sigma_i = l(c(t))\langle\overrightarrow{l_1 \cdots l_h}\rangle - l_i(c(t))\langle\overrightarrow{l_1 \cdots g \cdots l_h}\rangle_i \tag{4.8}$$

a section of the twisted bundle

$$c^*(T_{\mathbf{X}}(1)).$$

Let's define a quotient bundle. Because c is a birational map to its image, there are finitely many points $t_i \in \mathbf{P}^1$ where the differential map

$$c_* : T_{t_i}\mathbf{P}^1 \quad \to \quad T_{c(t_i)}\mathbf{P}^3 \tag{4.9}$$

is not injective. Assume its vanishing order at t_i is m_i . Let

$$m = \sum_i m_i. \tag{4.10}$$

Let $s \in H^0(\mathcal{O}_{\mathbf{P}^1}(m))$ such that

$$div(s) = \Sigma_i m_i t_i.$$

The sheaf morphism c_* is injective and induces a composed morphism ξ of sheaves

$$T_{\mathbf{P}^1} \quad \stackrel{c_*}{\to} \quad c^*(T_X) \quad \stackrel{\frac{1}{s(t)}}{\to} \quad c^*(T_X) \otimes \mathcal{O}_{\mathbf{P}^1}(-m), \tag{4.11}$$

where $c_*(T_{\mathbf{P}^1})$ in $c^*(T_X)$ is a sub-sheaf generated by the image of the differential map c_*. Notice that $c_*(T_{\mathbf{P}^1})$ restricted to an open set $\mathbf{P}^1 \backslash \{t_i\}$ is a line bundle. Taking a closure, we obtain a sub line bundle of $c^*(T_X)$, whose degree is $m+2$, and denoted by $\mathcal{L}$. Then the morphism ξ is injective bundle morphism (over $\mathbf{P}^1$). Let

$$N_m(1) = \frac{c^*(T_X) \otimes \mathcal{O}_{\mathbf{P}^1}(-m)}{\xi(T_{\mathbf{P}^1})} \otimes c^*(\mathcal{O}_{\mathbf{P}^3}(1)). \tag{4.12}$$

Then σ_i/s is reduced to a section of $N_m(1)$. By the adjunction formula

$$N_m(1) \simeq \mathcal{O}_{\mathbf{P}^1}\bigg((5-h)d - m - 2\bigg).$$

If $h \geq 5$, $(5-h)d - m - 2 < 0$. Hence $\frac{\sigma_i}{s}$ is reduced to zero in $N(1)$. Therefore it is a section of the line bundle $\xi(T_{\mathbf{P}^1}) \otimes c^*(\mathcal{O}_{\mathbf{P}^3}(1))$. The equations

$$l(c(t)) = l_i(c(t)) = 0, all\ i$$

have distinct hd zeros. Observing the expression (4.8), $\langle\overrightarrow{l_1 \cdots l_h}\rangle$ must lie in the sub-bundle

$$\mathcal{L} \subset c^*(T_X)$$

at these zeros which are smooth points of the regular map c.

Notice the bundle $c^*(T_{\mathbf{P}^3})$ is generated by global sections. So is

$$\frac{c^*(T_{\mathbf{P}^3})}{\mathcal{L}}.$$

Hence

$$\frac{c^*(T_{\mathbf{P}^3})}{\mathcal{L}} \simeq \mathcal{O}_{\mathbf{P}^1}(k_1) \oplus \mathcal{O}_{\mathbf{P}^1}(k_2), \tag{4.13}$$

where k_1, k_2 are non-negative. Since the degree of

$$\frac{c^*(T_{\mathbf{P}^3})}{\mathcal{L}}$$

is $4d - m - 2$. This implies

$$k_i \leq 4d - m - 2 < 4d.$$

Thus the section $\langle\overrightarrow{l_1 \cdots l_h}\rangle$ is a section of the sheaf $c_*(T_{\mathbf{P}^1})$. On the other hand, the derivative of f in the direction of $\langle\overrightarrow{l_1 \cdots l_h}\rangle$ is exactly $l_1 \cdots l_h|_{c(t)}$ which is non-zero. Hence the section $\langle\overrightarrow{l_1 \cdots l_h}\rangle$ does not lie in the bundle T_X. Therefore it can't be a section of $c_*(T_{\mathbf{P}^1})$. This is a contradiction. $\square$

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