Hilbert scheme of rational curves on a generic Fano hypersurface

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Abstract

Let $X_0 \subset \mathbf{P}^n$ be a generic hypersurface of degree h.

Let $R_d(X_0)$ denote an open set of the Hilbert scheme parameterizing irreducible rational curves of degree d on X_0. We prove that

(1) If $4 \leq h \leq n-1$, $R_d(X_0)$ is an integral, local complete intersection of dimension

$$(n+1-h)d + n - 4. \tag{0.1}$$

(2) If furthermore $(h^2-n)d+h \leq 0$ and $h \geq 4$, in addition to part (1), $R_d(X_0)$ is also rationally connected,

1 Introduction

Let $\mathbf{P}^n$ be projective space of dimension n over complex numbers $\mathbb{C}$. Let $X_0 \subset \mathbf{P}^n$ be a generic hypersurface of degree h. Let

$$R_d(X_0) \subset \{c : c \subset X_0\} \tag{1.1}$$

denote an open set of the Hilbert scheme parameterizing irreducible rational curves of degree d on X. This is a subscheme of the Hilbert scheme of rational curves of degree d in $\mathbf{P}^n$.

Key words: Generic hypersurfaces, Hilbert scheme, Jacobian matrix, Clemens' conjecture

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Theorem 1.1. *(Main theorem).*

(a) If $4 \leq h \leq n-1$*,* $R_d(X_0)$ *is an integral, local complete intersection of dimension*

$$(n+1-h)d+n-4. \tag{1.2}$$

(b) Furthermore, if $(h^2-n)d+h \leq 0$ *and* $h \geq 4$*,* $R_d(X_0)$ *is a rationally connected, integral, local complete intersection of dimension*

$$(n+1-h)d+n-4. \tag{1.3}$$

Reamrk Let $R_d(X_0)_s$ be the open set of $R_d(X_0)$ consisting of smooth curves. Then J, Harris, M. Roth and J. Starr's $R_d(X)$ had a similar result for $R_d(X_0)_s$.

1.1 Variant version

J, Harris, M. Roth and J. Starr had similar results earlier. They have studied Konsiverch's the moduli $\mathcal{M}_d$ of stable maps from $\mathbf{P}^1$. The result relies on the detailed knowledge of the Konverich's moduli space $\mathcal{M}_d$, especially its boundary. We use a different approach.

Our space of maps, I_{X_0} is not a moduli space. Thus we call it a variant version. We'll show that the variant version I_{X_0} is smooth and connected.

1.2 Outline of the proof

In string theory, there are two different theories, "non linear sigma model" and "gauged linear sigma model". Kontsevich's moduli space of stable maps is a starting point of the rigorous, mathematical theory for "non linear sigma model". Our research focus on the mathematical structures of fields in "gauged linear sigma model", which is also called a linear model of stable moduli in [4]. There is a filtration on this model which is helplessly simple on its own. However its interplay with hypersurfaces is non trivial. The reason to use the linear model is that, the incidence scheme of rational maps on generic hypersurface in the case of study, is a "mostly" smooth subscheme of a projective space. Once the scheme is smooth, everything else will follow automatically. The linear model has advantages and disadvatages when compairing with Kontsevich's moduli space of stable maps. Our general idea in [10], [11], [12] and this paper concentrates on the advantages which the linear model offers.

Let $S = \mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^n}(h)))$ be the space of all hypersurfaces of degree h.
Let

$$\mathbb{C}^{(n+1)(d+1)}$$

be the vector space,

$$(H^0(\mathcal{O}_{\mathbf{P}^1}(d))^{\oplus n+1}$$

whose open subset parametrizes the set of maps

$$\mathbf{P}^1 \to \mathbf{P}^n$$

whose push-forward cycles have degree d. [1] Throughout the paper, we let

$$M = \mathbb{C}^{(n+1)(d+1)}.$$

M has affine coordinates. The "gauged linear sigma model" uses the space M that has a stratification of closed subvarieties,

$$M = M_d \supset M_{d-1} \supset \cdots \supset M_0 = \{constant\ maps\}, \tag{1.4}$$

where

$$M_i = \{(gc_0, \cdots, gc_n) : g \in H^0(\mathcal{O}_{\mathbf{P}^1}(d-i)), c_j \in H^0(\mathcal{O}_{\mathbf{P}^1}(i))\}. \tag{1.5}$$

This stratification makes it impossible to view M as a space of morphisms of the same degree d, i.e. $M_d \neq Hom_d(\mathbf{P}^1, X)$.

Let

$$\Gamma$$

be the incidence scheme

$$\{(c, [f]) \subset M \times S : c^*(f) = f(c(t)) = 0\} \tag{1.6}$$

[1]The automorphism of $\mathbf{P}^1$ induces a $PGL(2)$ group action on $\mathbf{P}(\mathbb{C}^{(n+1)(d+1)})$. Let

$$PGL(2)(c_0) \subset \mathbf{P}(\mathbb{C}^{h(d+1)})$$

be the orbit of $c_0 \in \mathbf{P}(\mathbb{C}^{h(d+1)})$.

Let Γ_f be the projection of the fibre of Γ over f to M.

The natural dominant rational map,

$$\Gamma_f \stackrel{\mathcal{R}}{\dashrightarrow} R_d(X,h), \tag{1.7}$$

reduces theorem 1.1 to showing that Γ_f is a rationally connected, integral variety of the expected dimension. This rational map $\mathcal{R}$ is constructed and verified by the results in [9], I 6.6.1, II 2.7. We'll discuss the details of this in section 4.

Using this conversion, in the rest of the paper we concentrate on the scheme Γ_f. Notice Γ_f has an induced filtration

$$\Gamma_f \supset (M_{d-1} \cap \Gamma_f) \supset \cdots \supset (M_0 \cap \Gamma_f). \tag{1.8}$$

Notice by results in [9] (mentioned above) $\mathcal{R}$ is regular on the inverse of $R_d(X,h)$ because the rational curves in $R_d(X,h)$ are all irreducible. But it may not be regular on the lower stratum of (1.4).

Then theorem 1.1 follows from the propositions on Γ_f below .

Proposition 1.2. *If $4 \leq h \leq n$, then for each $d \geq 1$, the scheme*

$$\Gamma_f \backslash M_0 \tag{1.9}$$

is smooth.

Remark The scheme Γ_f is singular at the points in M_0.

Proposition 1.3. *If $4 \leq h \leq n-1$, then for each $d \geq 1$, the scheme*

$$\Gamma_f \backslash M_0 \tag{1.10}$$

is connected.

Remark When $h = n$ our method failed to prove the connectivity of Γ_f.

Proposition 1.4. *If $h \geq 4$ and $(h^2 - n)d + h \leq 0$, then the scheme*

$$\Gamma_f \backslash M_0 \tag{1.11}$$

is a rationally connected, integral, complete intersection of M defined

$$f(c(t_1)) = \cdots = f(c(t_{hd+1})) = 0, \tag{1.12}$$

where $t_1, \cdots, t_{hd+1}$ are any distinct points of $\mathbf{P}^1$.

The propositions (1.3), (1.4) follow from the proposition (1.2) which follows from a rather plausible, but difficult lemma

Lemma 1.5. *Let $\mathcal{G}$ be the Gauss map*

$$X \quad \to (\mathbf{P}^n)^*. \tag{1.13}$$

Let $c : \mathbf{P}^1 \to X$ be a non-constant regular map (with an image of any degree). Assume X is generic and $h \geq 4$. Then for generic

$$(t_1, \cdots, t_h) \in Sym^h(\mathbf{P}^1),$$

$$\mathcal{G}(c(t_1)), \cdots, \mathcal{G}(c(t_h))$$

are linearly independent.

2 Smoothness of the linear model

Lemma 1.5 is the key to the results. Its proof lies in the heart of one difficult question that is essential to many important problems in this area. In this paper we would not explore this difficult question, but refer it to the complete papers [10], [11], and [12]. Let's prove lemma 1.5.

Proof. of lemma 1.5: We prove it by a contradiction. Suppose there are a generic hypersurface $X_0 = div(f_0)$ of degree h, a non-constant rational map $c_0 : \mathbf{P}^1 \to X_0$, birational to its image, and h points

$$c_0(t_1), \cdots, c_0(t_h)$$

such that

$$\mathcal{G}(c_0(t_1)), \cdots, \mathcal{G}(c_0(t_h))$$

are linearly dependent. Then

$$dim(\mathcal{G}(c_0(t_1)) \cap \cdots \cap \mathcal{G}(c_0(t_h))) \geq n - h + 1 \tag{2.1}$$

and for any vector $\alpha \in \mathcal{G}(c_0(t_1)) \cap \cdots \cap \mathcal{G}(c_0(t_h),$

$$\frac{\partial f_0}{\partial \alpha}|_{c_0(t)} = 0, \tag{2.2}$$

for all $t \in \mathbf{P}^1$. Let $\{\alpha_j, j = 1, \cdots, r = n-h\}$ be a set of linearly independent vectors in

$$\mathcal{G}(c_0(t_1)) \cap \cdots \cap \mathcal{G}(c_0(t_h)))$$

Then c_0 lies on the hypersurfaces

$$\frac{\partial f_0}{\partial \alpha_j}|_{c_0(t)} = 0, j = 1, \cdots, r. \tag{2.3}$$

(Notice $f_0, \frac{\partial f_0}{\partial \alpha_j}$ are generic in the moduli of hypersurfaces). Hence it lies on the complete intersection

$$Y = \cap_j \{\frac{\partial f_0}{\partial \alpha_j} = 0\} \cap X_0. \tag{2.4}$$

By our assumption $h \geq 4$, we obtain that

$$dim(Y) = h - 1 \geq 3. \tag{2.5}$$

Next we are going to apply theorem 1.1 in [12]. We should elaborate the requirements for the theorem. Let's denote the sequence of hypersurfaces defining the complete intersection Y by

$$f_0, f_1 = \frac{\partial f_0}{\partial \alpha_1}, \cdots, f_r = \frac{\partial f_0}{\partial \alpha_r}. \tag{2.6}$$

There are two requirements for the proof of theorem 1.1 of [12]:

(1) the subvariety defined by $f_0 = \cdots = f_j = 0, j \leq r$ is smooth at $c_0(\mathbf{P}^1)$;

(2) each $f_j, j = 1, \cdots, r$ is a generic hypersurface. This is different from the actual notion "generic complete intersection" which usually means that the point

$$\begin{gathered}(f_1, \cdots, f_r) \in \\ H^0(\mathcal{O}_{\mathbf{P}^n}(h)) \times H^0(\mathcal{O}_{\mathbf{P}^n}(h-1)) \times \cdots \times H^0(\mathcal{O}_{\mathbf{P}^n}(h-1))\end{gathered} \tag{2.7}$$

is generic.

These two conditions are satisfied because in our case f_0 is generic. Therefore by the theorem 1.1 in [12],

$$H^1(N_{c_0/Y}) = 0. \tag{2.8}$$

Next we apply $H^1(N_{c_0/Y}) = 0$ to deduce an inequality. First let

$$c_0^*(T_Y) = \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbf{P}^1}(a_{dim(Y)}). \tag{2.9}$$

Because $H^1(N_{c_0/Y}) = 0$,

$$a_j \geq -1, j = 1, \cdots, dim(Y). \tag{2.10}$$

Because at least one a_j is larger than or equal to 2 (from automorphisms of $\mathbf{P}^1$), we obtain that

$$c_1(c_0^*(T_Y)) \geq -dim(Y) + 3 = -h + 4. \tag{2.11}$$

Now we use adjunction formula to find

$$c_1(c_0^*(T_Y)) = [n + 1 - (h + (n-h)(h-1))]d. \tag{2.12}$$

Now we apply the inequality $h \leq n - 1$ to obtain that

$$c_1(c_0^*(T_Y)) \leq -h + 2 \tag{2.13}$$

Then (2.11) becomes

$$-h + 2 \geq -h + 4. \tag{2.14}$$

This is absurd. Therefore c_0 does not exist. The lemma 1.5 is proved.

□

Next we prove proposition 1.2:

Proof. We consider a $c_0 \in \Gamma_f \backslash M_0$. The idea of the proof is similar to that in [10] or [11]. We are going to choose affine coordinates for M and defining equations for Γ_f. Then use them to calculate the Jacobian matrix of Γ_f. Let's start with coordinates of M.

Let $t_1, t_2, \cdots, t_h$ be the points in lemma 1.5. Choose affine coordinates

$$z_0, \cdots, z_n$$

$\mathbb{C}^{n+1}$ such that $\{z_i = 0\}$ for $i = 1, \cdots, h$ are exactly $\mathcal{G}(c_0(t_i))$. Next we choose affine coordinates for M. In each copy $H^0(\mathcal{O}_{\mathbf{P}^1}(d)$ of M, we express

$$c_j(t) = \sum_{k=0}^{d} c_j^k t^k \in H^0(\mathcal{O}_{\mathbf{P}^1}(d))$$

(for j-th copy) as

$$c_j(t) = \sum_{k=0}^{d} \theta_j^k (t - t_j)^k. \tag{2.15}$$

where t_j for $j = 1, \cdots, h$ are the those in lemma 1.5, and $t_j = 0$ if j is not in the interval $[1, h]$. The θ_j^k are affine coordinates for M. We would like to use coordinates w_j^k satisfying (a linear transformation of θ_j^k)

$$\begin{cases} w_j^k = \theta_j^k, & k \neq 0 \\ w_j^0 = \sum_{k=0}^{d} \theta_j^k (t' - t_j)^k. & \end{cases} \tag{2.16}$$

where t' is a generic complex number.

Let the corresponding coordinates for the point c_0 be $\bar{\theta}_j^k$. Next we choose defining equations of Γ_f at c_0. Consider the following homogeneous polynomials in W_j^k.

$$\begin{cases} f(c(t')) = \sum_{j=0}^{n} \epsilon_j W_j^0 & \\ \frac{\partial^j f(c(t_1))}{\partial t^j} & j = 1, \cdots, d \\ \cdots & \\ \frac{\partial^j f(c(t_h))}{\partial t^j} & j = 1, \cdots, d \end{cases} \tag{2.17}$$

Notice that, besides the first equation, the rest of them are the coefficients of the terms between the first order and d-th order in the Taylor expansion of $f(c(t))$ in t around the points $t_1, \cdots, t_h$. We claim that these polynomials define the scheme Γ_f.

To see this, we let c be a point in the scheme defined by the polynomials in (2.17). Also let

$$f(c(t)) = \sum_{i=0}^{hd} \mathcal{K}_i(c)t^i. \tag{2.18}$$

Using an automorphism of $\mathbf{P}^1$, we may assume $t_1 = 0$. Then the equations

$$\frac{\partial^j f(c(t_1))}{\partial t^j} = 0, j = 1, \cdots, d$$

imply that

$$\mathcal{K}_i = 0, 1 \leq i \leq d. \tag{2.19}$$

Then $f(c(t))$ satisfying the first set of equations

$$\frac{\partial^j f(c(t_1))}{\partial t^j} = 0, j = 1, \cdots, d$$

becomes

$$f(c(t)) = \mathcal{K}_0(c) + r^d(\sum_{i=1}^{d(h-1)} \mathcal{K}_{d+1+i}(c)t^i). \tag{2.20}$$

Next we repeat the same process inductively for the term

$$\sum_{i=1}^{d(h-1)} \mathcal{K}_{d+1+i}(c)t^i$$

to obtain all $\mathcal{K}_i = 0, i \geq 1$. At last $\mathcal{K}_0 = 0$ because $f(c(t')) = 0$. Hence $c \in \Gamma_f$. To prove the proposition it suffices to show that the Jacobian matrix of

$$f(c(t')) = 0, \frac{\partial^j f(c(t_1))}{\partial t^j} = 0, \frac{\partial^j f(c(t_h))}{\partial t^j} = 0$$

with respect to the variables w_j^k has full rank. (Note w_j^k are the coordinates for c). Since we use some "liberality" of w_j^k, we continue the proof using the affine coordinates w_i^k directly.

Consider the subspace V of M defined by θ_0^k, θ_l^k are equal to $\bar{\theta}_0^k, \bar{\theta}_l^k$ where $k \neq 0$ and l is not one of $1, \cdots, h$. Then the scheme $\Gamma_f \cap V$ is defined by

$$\begin{cases} f(0, \sum_{k=0}^d \theta_1^k (t-t_1)^k, \cdots, \sum_{k=0}^d \theta_h^k (t-t_h)^k, 0, \cdots, 0) = 0 \\ \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \mathit{for\ all}\ t \end{cases} \tag{2.21}$$

We start this with h equations in (2.17) that are first order partial derivatives, i.e. the coefficients of $t-t_1, \cdots, t-t_h$ are zero. Because the lemma 1.5, They are equivalent to the equations

$$\frac{\partial f(c_0(t_1))}{\partial z_1}\theta_1^1 = \cdots = \frac{\partial f(c_0(t_h))}{\partial z_h}\theta_h^1 = 0 \tag{2.22}$$

where

$$z_1, \cdots, z_h \tag{2.23}$$

are variables for the affine coordinates of $\mathbb{C}^{n+1}$. By the lemma 1.5, we know that

$$\theta_1^1 = \cdots = \theta_h^1 = 0.$$

Next step is to consider another h equations in second derivatives. Notice that the general curves become

$$(0, \sum_{k\neq 1}\theta_1^k(t-t_1)^k, \cdots, \sum_{k\neq 1}\theta_h^k(t-t_h)^k, 0, \cdots, 0) \tag{2.24}$$

i.e., there are no linear terms.

$$\frac{\partial f(c_0(t_1))}{\partial \alpha_1^2} = \cdots = \frac{\partial f(c_0(t_h))}{\partial \alpha_h^2} = 0. \tag{2.25}$$

Then using the lemma 1.5 to obtain that

$$\alpha_1^2 = \cdots = \alpha_h^2 = 0. \tag{2.26}$$

Recursively we obtain that the solution to the system of linear equations (2.21) is all $\alpha_j^k, j=1,\cdots,h, k=0,\cdots,d$ satisfying

$$\begin{array}{c} \sum_{i=0}^{n}\frac{\partial f(c_0(t'))}{\partial \alpha_j^0} = 0 \\ \alpha_j^k = 0, j=1,\cdots,h, k=1,\cdots,d. \end{array} \tag{2.27}$$

This means that the set of solutions to the equations (2.21) has dimension $h-1$. Thus the rank of Jacobian matrix of Γ_f at c_0 is $hd+1$, i.e. it has full rank. Hence Γ_f is smooth at c_0 whenever c_0 is a non-constant.

This completes the proof.

□

2.1 Connectivity

This section will prove proposition 1.3. In last section we proved that

$$\Gamma_f \backslash M_0$$

is a smooth variety of dimension

$$(n+1)(d+1) - (hd+1).$$

To show it is irreducible, it suffices to show it is connected. Let Γ'_f be an irreducible component of Γ_f. Assume $d \geq 2$.

Then

$$dim(\Gamma'_f) \geq (n+1)(d+1) - (hd+1) = (n+1-h)d + n \tag{2.28}$$

Let

$$M^{d-1} = \mathcal{O}_{\mathbf{P}^1}(d-1)^{\oplus n+1}. \tag{2.29}$$

We should note that $M_{d-1} \simeq \mathbb{C} \times M^{d-1}$. It has a similar stratification

$$M^{d-1} = M^{d-1}_{d-1} \supset M^{d-1}_{d-2} \supset \cdots \supset M^{d-1}_0 = \{constant\ maps\}, \tag{2.30}$$

where

$$M^{d-1}_i = \{(gc_0, \cdots, gc_n) : g \in H^0(\mathcal{O}_{\mathbf{P}^1}(d-1-i)), c_j \in H^0(\mathcal{O}_{\mathbf{P}^1}(i))\}.$$

Notice

$$dim(\Gamma'_f) = (n+1-h)d + n$$

and

$$dim(M_{d-1}) = (n+1)d + 2.$$

Hence $\Gamma'_f \cap M_{d-1}$ is non-empty and Then every irreducible components of $\Gamma'_f \cap M_{d-1}$ is birational (not isomorphic) to an irreducible component of

$$\mathbb{C} \times \Gamma^{d-1}_f, \tag{2.31}$$

where Γ^{d-1}_f is defined to be

$$\{c \in M^{d-1} : c \subset f\}, \tag{2.32}$$

and $\mathbb{C}$ is an affine open set of $\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^1}(1)))$. Notice

$$\begin{aligned} dim(\Gamma_f^{d-1} \cap M_0^{d-1}) = d+n-1 \\ dim(\Gamma_f^{d-1}) \geq (n+1-h)(d-1)+n \end{aligned} \tag{2.33}$$

Because $h \leq n-1$, $d \geq 2$,

$$dim(\Gamma_f^{d-1}) > dim(\Gamma_f^{d-1} \cap M_0^{d-1}). \tag{2.34}$$

Therefore every component of $\Gamma'_f \cap M_{d-1}$ contains a non-constant c.

Thus inside smooth locus of Γ_f, every point is connected to a point in the lower stratum. Then by the induction it suffices to prove that the second lowest stratum in $\Gamma_f \cap (M_1 \backslash M_0)$, which consists of all maps that correspond to lines, is connected. By the classical result of Fano variety of lines, this is correct. More precisely

$$\Gamma_f \cap (M_1 \backslash M_0)$$

is birational to

$$\mathbb{C}^{d-1} \times \Gamma_f^1$$

where Γ_f^1 is the same as (2.31) with $d = 2$, and $\mathbb{C}^{d-1}$ is an affine open set of

$$\mathbf{P}(H^0(\mathcal{O}_{\mathbf{P}^1}(d-1))).$$

Then it suffices to prove

$$\Gamma_f^1$$

is irreducible. The image of Γ_f^1 under the rational map $\mathcal{R}$ is just an open set of Fano variety $F(X)$ of lines on the generic hypersurface $X = \{f = 0\}$. It is connected by the classical result (see theorem 4.3, [9]). Therefore the proposition 1.3 is proved.

3 Rationally connectedness

Proof. Note

$$\mathbf{P}(M_0) \tag{3.1}$$

is a smooth subvariety of $\mathbf{P}^{(n+1)d+n}$, with dimension

$$d+n-1.$$

Choose two generic planes V_{top}, V_{bott} in $\mathbf{P}^{(n+1)d+n}$ with dimensions

$$nd-1, n+d$$

respectively. Consider the dominant projection map

$$\Gamma_f \backslash (\Gamma_f \cap V_{top}) \quad \to \quad V_{bott}. \tag{3.2}$$

Because $(h^2-n)d+h \leq 0$, the fibre's dimension is at least

$$(n-h)d-1$$

which is larger than or equal to one. By Bertini's theorem, the generic fibre is a smooth complete intersection of $hd+1$ hypersurfaces of degree h followed by $n+d$ many hyperplanes in a projective space of dimension $(n+1)d+n$.

Notice the generic fibre satisfies

$$h(hd+1)+n+d \leq (n+1)d+n. \tag{3.3}$$

(because $(h^2-n)d+h \leq 0$), where the left hand side is the sum of the degrees of all hypersurfaces and right hand side is the dimension of the projective space. Hence the generic fibre is a smooth Fano variety. By V (2.1) and (2.13) of [9], it is rationally connected. By corollary 1.3, [5],

$$\Gamma_f \backslash (\Gamma_f \cap V_{top}) \tag{3.4}$$

must be rationally connected. The proof is completed.

□

To summarize it, we just proved that

Theorem 3.1.

(1) If $4 \leq h \leq n-1$, Γ_f is an integral, complete intersection of dimension

$$(n+1-h)d+n. \tag{3.5}$$

(2) Furthermore, if $(h^2-n)d+h \leq 0$ and $h \geq 4$, Γ_f is a rationally connected, integral, complete intersection of dimension

$$(n+1-h)d+n. \tag{3.6}$$

Proof. of theorem 1.1.: Next we show that the results of theorem 3.1 also hold for the open set $R_d(X,h)$ of Hilbert scheme. Let $c_{bi} \in \Gamma_f$ be a point of Γ_f such that $c_{bi}, \mathbf{P}^1 \to X$ is birational to its image. There is a rational map map

$$\begin{array}{ccc} \Gamma_f & \stackrel{\mathcal{R}_1}{\dashrightarrow} & Hom_{bir}(X)^{sn} \\ (c_0, \cdots, c_n) & \to & graph(\{t\} \to [c_0(t), \cdots, c_n(t)]) \end{array} \tag{3.7}$$

where sn stands for semi-normalization. Next we use the results from [9], namely I theorem 6.3, II comment 2.7. to construct the composition in a neighborhood of c_{bi},

$$Hom_{bir}(X)^{sn} \ \to \ CH(W) \ \to Hilb(X)^{sn} \tag{3.8}$$

Finally $\mathcal{R}$ is defined to be the composition in a neigborhood

$$G_f \ \stackrel{\mathcal{R}_1}{\to} \ Hom_{bir}(X)^{sn} \ \stackrel{\mathcal{R}_2}{\to} \ CH(W) \ \stackrel{\mathcal{R}_3}{\to} Hilb(X)^{sn}. \tag{3.9}$$

By propsoition 1.3, c_{bi} is a smooth point of Γ_f. Then $Hom_{bir}(X)^{sn}$ is normal at c_{bi}. Then the map $\mathcal{R}$ is regular at c_{bi} because $\mathcal{R}_3$ is an isomorphism by I theorem 6.3, [9], $\mathcal{R}_1$ is a smooth map with the fibres of dimension 1 by the argument for (2.15), [10], and $\mathcal{R}_2$ is a smooth map with fibres of dimension 3 by II 2.7, [9]. Then theorem 1.1 follows from theorem 3.1

□

4 Work of Harris et al.

Our main theorem extends the current known results in this area. Let $R_d(X)$ be the open set of the Hilbert scheme parametrizing smooth, irreducible rational curves of degree d. One should notice $R_d(X) \neq R_d(X,h)$. In [6] and [7], J. Harris, J. Starr et al proved that

Theorem 4.1. *(Harris, Starr et al).*

(1) If $h < \frac{n+1}{2}, n \geq 3$, $R_d(X)$ is an integral, locally complete intersection of the expected dimension

$$(n+1-h)d + n - 4.$$

(2) If furthermore $h \leq \frac{-1+\sqrt{4n-3}}{2}$ and $n \geq 3$, in addition to that in part (1), $R_d(X)$ is also rationally connected.

The part (1) which is in their first paper, is furthered by Coskun, Beheshti, Kumar and many others ([1], [3], etc). It is conjectured by Coskun and Starr ([3]) that if $h \leq n$, $R_d(X)$ is irreducible, and has the expected dimension

$$(n + 1 - h)d + n - 4.$$

Theorem 4.2. *Coskun and Starr's conjecture is correct for* $4 \leq h \leq n - 1$.

Proof. This is theorem 1.1, part (1). □

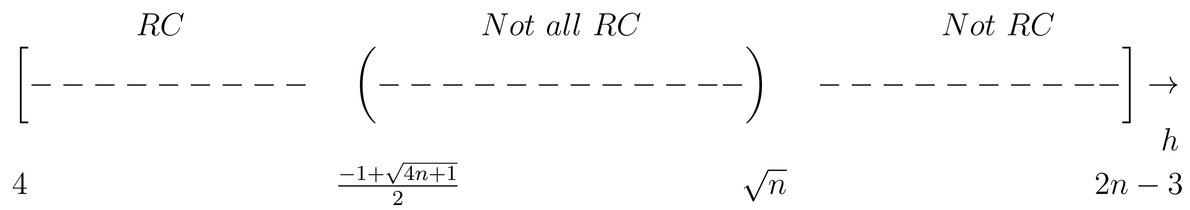
Remark The cases $h = 1, 2$ are known by the result of Kim and Pandharipande ([8]). The case $h = 3$ is also completely solved in [3].

The following lemma deals with the rational connectedness.

Corollary 4.3.
(1) If $h \leq \frac{-1+\sqrt{4n+1}}{2}$ *and* $h \geq 5$, *then for each degree d, the Hilbert scheme* $R_d(X, h)$ *is a rationally connected, integral, local complete intersection of the expected dimension.*

(2) If $\frac{-1+\sqrt{4n+1}}{2} < h < \sqrt{n}$ *and* $h \geq 5$, *then for each degree* $d \geq \frac{h}{n-h^2}$, $R_d(X, h)$ *is a rationally connected, integral, local complete intersection of the expected dimension.*

Remark The part (1) of the corollary improves Harris, Roth and Starr's bound by a little. Part (2) reveals something new which says $R_d(X, h)$ will not immediately becomes non-rationally connected as the degree of hypersurface increases. The range $(\frac{-1+\sqrt{4n+1}}{2}, \sqrt{n})$ for h serves as a "buffer-zone" for the rational connectedness to fadeout. The following is the conjectural graph of such a distribution



where RC stands for rationally connected. In the graph this paper proves all RC statements, but did not prove any of non RC statements for which we only know a handful of indirect examples.

For the irreducibility this "buffer zone" may only consists of one number. See section 5.

Proof. If $h \leq \frac{-1+\sqrt{4n+1}}{2}$, $h^2 + h - n \leq 0$. Hence $h^2 - n < 0$. Then

$$(h^2 - n)d + h < 0 \tag{4.1}$$

holds for all $d \geq 1$. Then by Main theorem 1.1, the part (1) is proved.

The part (2) of the corollary is just the part (2) of Main theorem 1.1.

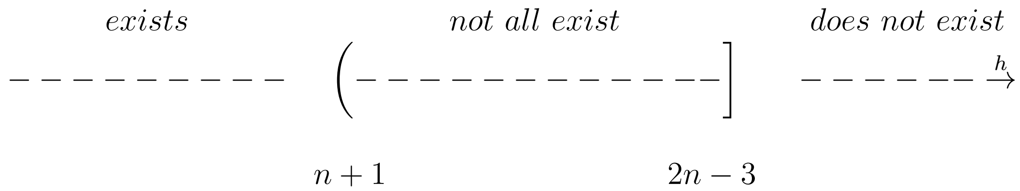
□

5 Hilbert scheme of rational curves

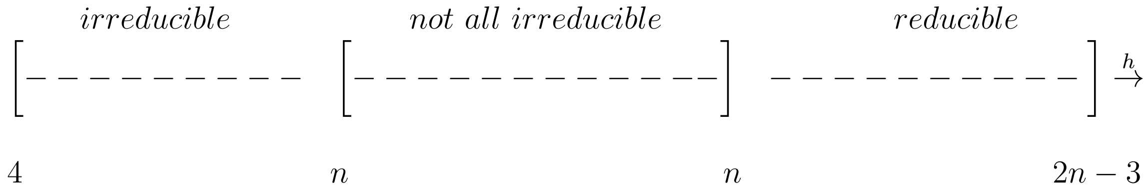
In this section we would like to organize our results in the area of rational curves on hypersurfaces. This extends to hypersurfaces in other two categories: Calabi-Yau, of general type. As before let $X \subset \mathbf{P}^n$ be a generic hypersurface of degree h. Let $R_d(X, h)$ denote an open set of the Hilbert scheme parameterizing irreducible rational curves of degree d on X.

In general the full scheme structure of $R_d(X, h)$ depends on the full scheme structure of X. But some of basic structures of $R_d(X, h)$ only depend on the indices d, h and n. We would like to discuss these structures shared by "almost all (generic)" hypersurfaces. In the following we describe three basic structures of Hilbert scheme $R_d(X, h)$: (1) existence, (2) irreducibility, (3) rational connectivity.

(1) For the existence, we have the following demographic picture. Its correctness was proved in ([11])



(2) For the irreducibility, we have the following demographic picture for $h \geq 4$,



The the statements for $h \geq n$ are our conjectures.

(3) For the rational connectivity, we have the conjectural demographic picture in the last section.

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